LINEAR ALGEBRA HOMEWORK 7

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Exercise 0.1. *Decide if* $A = [e_3, e_1 + e_2, e_2] \in M_3$ *is invertible. If so, compute* A^{-1}.

Exercise 0.2. *As we have sketched in class, you will fill in the proof that there is a bijection between the set of* $GL_n(F)$*-conjugation classes in* $M_n(F)$ *, and the set of isomorphism classes of* $F[t]$*-spaces* V *of* $\dim_F V = n$*. To each matrix* X*, define the* $F[t]$*-space by the* F*-algebra map*

$$\varphi_X : F[t] \to \operatorname{End} F^n \equiv M_n(F), \quad t \mapsto X.$$

Argue if $g \in GL_n$*, then* $\varphi_{gXg^{-1}}$ *defines an isomorphic* $F[t]$*-space. Verify that the correspondence* $[X] \mapsto [\varphi_X]$ *is a bijection from conjugation classes of matrices to isomorphism classes of* $F[t]$*-spaces.*

Exercise 0.3. WRITE UP *For* $X \in M_n$*, put* $k(X) := \dim \ker X$*. Assume* $X^2 = 0$.
(a) Show that $k(X) \geq n/2$.
In less than 1 page, show that the following:
(a) A conjugation class $[X]$ *in* $\operatorname{sol}(X^2 = 0)$ *in* M_n *is uniquely determined by* $k(X)$.
(b) Given any integer $k \geq n/2$*, there is a unique conjugation class* $[X]$ *of such solutions such that* $k(X) = k$.
After doing this right, you will be quite close to Project 1.

Exercise 0.4. WRITE UP *Let* $U \subset V$ *be a subspace, and let* $X \in \operatorname{End} V$*. We say that* U *is* X *invariant if* $XU \subset U$*. In 5 lines, prove that the induced map*

$$\bar{X} : V/U \to V/U, \quad v + U \mapsto Xv + U$$

satisfies the following: if $p(t) \in F[t]$ *and if* $p(X) = 0$ *in* $\operatorname{End} V$*, then* $p(\bar{X}) = 0$ *in* $\operatorname{End} V/U$.