- *Project* 1: Sufficient condition for constraint optimization in $\mathbb{R}^n$.

  In class we have derived the necessary condition, that is, the existence of Lagrange multipliers for a local minimum. In this project we will try to derive a sufficient condition for a local minimum. More precisely, consider the problem

  $$\text{Optimize } f(\mathbf{x}) \text{ subject to } g_1(\mathbf{x}) = g_2(\mathbf{x}) = \cdots = g_m(\mathbf{x}) = 0$$

  where $f, g_1, \cdots, g_m$ are assumed to be C^3 functions defined on some region $E \subset \mathbb{R}^n$. Let $F = f + \sum_{j=1}^{m} \lambda_j g_j$ where λ_j, $j = 1, \cdots, m$ are the Lagrange multipliers. From solving the equation $\nabla F(\mathbf{x}) = 0$ we can find the critical points of F, which are candidates of local minimums. We call a critical point $\overline{\mathbf{x}}$ <u>*non-degenerate*</u> if it satisfies

  $$\det \begin{bmatrix} \nabla^2 F(\overline{\mathbf{x}}) & B(\overline{\mathbf{x}}) \\ B(\overline{\mathbf{x}})^T & \mathbf{0} \end{bmatrix} \neq 0$$

  where B is the $n \times m$ matrix given by

  $$B = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \frac{\partial g_2}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}$$

  Try to show that if a non-degenerate critical point $\overline{\mathbf{x}}$ satisfies $\mathbf{h}^T \nabla^2 F(\overline{\mathbf{x}})\mathbf{h} \geqslant 0$ for all nonzero $\mathbf{h}$ such that $B^T(\overline{\mathbf{x}})\mathbf{h} = \mathbf{0}$, then the constraint optimization problem has a local minimum at $\overline{\mathbf{x}}$.

- *Project* 2: Necessary condition for a strong local minimum of the fixed endpoint problem.

  Recall the fixed endpoint problem is to minimize $\int_a^b f(x, y, y')dx$ over curves y that satisfy the endpoint conditions $y(a) = y_0$, $y(b) = y_1$ for some $y_0, y_1 \in \mathbb{R}$. A *piecewise differentiable* curve (i.e., a differentiable curve with only finite number of corner points) $y = \overline{y}(x)$ is a *strong local minimum* of J if there exists $\epsilon > 0$ such that $J(\overline{y}) \leqslant J(y)$ for all y that satisfies $|y(x) - \overline{y}(x)| < \epsilon$, $\forall x \in [a, b]$.

  Now for the given function $f = f(x, y, z)$, we define the <u>*Weierstrass E-function*</u> as

  $$E(x, y, z, w) := f(x, y, w) - f(x, y, z) - (w - z) f_z(x, y, z).$$

  Try to show that if $\overline{y}$ is a strong local minimum of the fixed endpoint problem, then we must have

  $$E(x, \overline{y}(x), \overline{y}'(x), w) \geqslant 0, \quad \forall \text{non-corner points } x \in [a, b] \text{ and } \forall w \in \mathbb{R}.$$

- *Project* 3: Variational approach to prove the Pontryagin's maximum principle.

  In general the proof of the Pontryagin's maximum principle is complicated and requires more knowledge than we covered in class. However, for certain special cases, it is possible to prove (part of) the maximum principle by the *variational* approach that we have seen throughout the class. This project is to explore such an approach. To be more precise, consider the *fixed time, free endpoint* optimal control problem:

  Control the system governed by the ordinary differential equation

  $$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$$

  from $\mathbf{x}_0 \in \mathbb{R}^n$ at initial time t_0 to any $\mathbf{x}_1 \in \mathbb{R}^n$ at a fixed time t_1, while the cost functional

  $$J(\mathbf{u}) = \int_{t_0}^{t_1} f_0(\mathbf{x}, \mathbf{u})\ dt$$

  is being minimized.

  In this project we will try to prove the following:

  Theorem: If $\mathbf{u} = \mathbf{u}^*(t)$ is an optimal control (meaning $J(\mathbf{u}^*) \leqslant J(\mathbf{u})$ for other piecewise continuous controls $\mathbf{u}$) and $\mathbf{x} = \mathbf{x}^*(t)$ is the corresponding trajectory that transfers $\mathbf{x}_0$ at t_0 to $\mathbf{x}_1$ at t_1, then there exists $\mathbf{p} = \mathbf{p}^*(t)$ (called *adjoint vector*) such that

  1. $\mathbf{x}^*, \mathbf{p}^*$ satisfy the *canonical* equations

     $$\begin{cases} \dot{\mathbf{x}}^* = H_{\mathbf{p}}(\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*) \\ \dot{\mathbf{p}}^* = -H_{\mathbf{x}}(\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*) \end{cases}$$

     with conditions $\mathbf{x}^*(t_0) = \mathbf{x}_0$, $\mathbf{p}^*(t_1) = \mathbf{0}$ and the function $H : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ defined by

     $$H(\mathbf{x}, \mathbf{u}, \mathbf{p}) = \mathbf{p} \cdot f(\mathbf{x}, \mathbf{u}) - f_0(\mathbf{x}, \mathbf{u}).$$

  2. For each fixed t, $\mathbf{u}^*$ is a critical point of the function $H(\mathbf{x}^*, \cdot, \mathbf{p}^*)$, namely,

     $$H_{\mathbf{u}}(\mathbf{x}^*, \mathbf{u}^*, \mathbf{p}^*) = 0,\ \forall t \in [t_0, t_1].$$