

Algebraic combinatorics final project problems

July 25, 2017

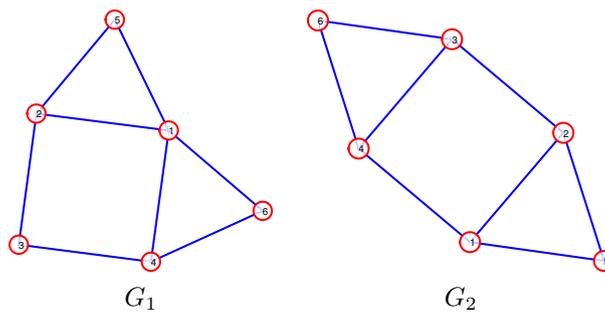
1 Determinant from m-set.

Let G be a finite m-set with n elements. List all the elements of G as g_1, g_2, \dots, g_n . For each g_i , we associate a variable x_{g_i} . Next, we form an $n \times n$ matrix M whose matrix elements are given by these variables, with $M_{ij} := x_{g_i g_j}$.

1. Calculate $\det(M)$ for a few small G .
2. For each of the examples, is it true that $\det(M) = \prod_{i=1}^c Q_i^{r_i}$, with each Q_i being an irreducible polynomial over \mathbb{C} , where c is the number of similarity classes of G , and $r_i = \deg(Q_i)$?
3. Try to prove what you found.

2 Graph Laplacian and quadratic forms.

Consider the following pair of graphs Let L_1, L_2 denote the Laplacian of the first and the second graphs,



respectively.

1. Show that the equivalence class of the integral quadratic form defined by the Laplacian, depends only on the isomorphism class of the graph.
2. Show that L_1 and L_2 are equivalent as integral quadratic forms.
3. Can you make a conjecture regarding when two graphs have equivalent integral quadratic forms? And conversely, if two graphs have equivalent integral quadratic forms, how are the two graphs related to each other?
4. Try to prove your conjectures.

3 An inequality.

1. Let G be a finite m-set with n elements. Denote by i the index of the commutator: i.e. $i = |G : [G, G]|$, and let c be the number of similarity classes. Prove that $4c - 3i \leq n$.
2. In addition, suppose G is a non-commutative simple m-set, show that we have the stronger inequality $9c \leq n$.
3. Can you further generalize the results, or say more in this direction?

4 Transportation generators of S_n

Given an arbitrary subset of the set of all transportations of S_n , count the number of minimal sets of generators of S_n contained in the subset. Namely, construct your best algorithm that computes this number, given any subset of the set of transportations as an input.

Terminology: by a minimal set of generators of S_n , we mean a set of generators of S_n , such that if any element gets deleted from the set, the remaining elements do not generate S_n .