Algebraic combinatorics I final project problems

July 28, 2019

1 System of equations on trees.

Let T_D denote the infinite regular tree graph with vertex valence $D \geq 2$. For each $ij \in E(T_D)$ an edge, we assign a real variable called edge length $a_{ij} > 0$.

Notation: $i \sim j$ means vertexes $i, j \in V(T_q)$ are neighbors, and

$$c_i = \sum_{j \sim i} \frac{1}{a_{ij}}, d_i = \sum_{j \sim i} \frac{1}{a_{ij}^2}.$$

We consider the following system of equations:

For each edge $ij \in E(T_D)$,

$$\frac{1}{a_{ij}}(\frac{c_i^2}{d_i^2} + \frac{c_j^2}{d_j^2}) - \frac{c_i}{d_i} - \frac{c_j}{d_j} = 0. \tag{1}$$

1. Fix a vertex V of T_D, we impose the condition that $\lim_{d(ij,V)\to\infty} a_{ij}$ exists (possibly infinite), where $d(ij, V)$ is the minimum of graph step distances between vertexes: $\min\{d(i, V), d(j, V)\}$, where $d(a, b)$ denotes the minimum number of steps to go from vertex a to vertex b. Note that this condition is independent of the choice of V. Find all solutions to the above system of equations (1) under this condition, and prove your assertion.
2. Is it possible to generalize your result above to more general graphs?
3. Without the above limit condition, can you find some more interesting classes of solutions?

2 Counting solutions over finite fields.

Let $f(x^I) = \sum_I a_I x^I$, where $I = (i_1, ..., i_n)$ runs over all possible combinations of nonnegative exponents $i_1, ..., i_n$ such that the total sum equals n. a_I are coefficients of x^I. Let $M_p = M_p(a_I)$ denote the number of solutions of $f(x^I) = 0$ over the finite field $\mathbb{F}_p$.

1. Find a formula for $M_p(\mod p)$ as a function of a_I.
2. When $n = 2$, rename the indexes and denote $f = a_1x_1^2 + a_0x_1x_2 + a_2x_2^2$. Let $g = (1 - \frac{4a_1a_2}{a_0^2})^{-\frac{1}{2}}$. When $a_0 \neq 0$, g can be regarded as a function $g(\frac{a_1}{a_0}, \frac{a_2}{a_0})$ of $\frac{a_1}{a_0}, \frac{a_2}{a_0}$. Explore the relation between $M_p(\mod p)$ and the Taylor series expansion of $g(\frac{a_1}{a_0}, \frac{a_2}{a_0})$ at $\frac{a_1}{a_0} = \frac{a_2}{a_0} = 0$, i.e. at $a_1 = a_2 = 0, a_0 = 1$, and prove your assertion.
3. Can you generalize the above relation to general n?

3 Invariants of a representation.

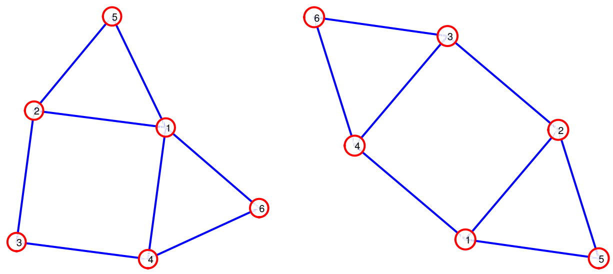
1. Let $G = \{n \times n$ matrices A whose entries are in the finite field $\mathbb{F}_2 \mid \det(A) = 1\}$. Then G is a finite group under multiplication.

   Multiplying on length n column vectors gives a representation ρ of G on a n-dimensional vector space V over $\mathbb{F}_2$ with standard basis $< x_1, ..., x_n >$. Now let W denote the vector space over $\mathbb{F}_2$ spanned by the basis given by all monomials in $x_1, ..., x_n$ of degree n. Then ρ naturally gives rise to a representation ρ_W of G on W. Let ρ_W^* denote the dual representation on the dual space W^*. Find a nonzero vector a in W^* that is an invariant under the ρ_W^* representation of G. i.e. a nonzero vector a such that $\rho_W^*(g)(a) = a$ for any $g \in G$. Prove your assertion.

2. Can you generalize the above result in some way?

4 Graph Laplacian and quadratic forms.

Consider the following pair of graphs. Let L_1, L_2 denote the Laplacian of the first and the second



G_1 G_2

graphs, respectively.

1. Show that the equivalence class of the integral quadratic form defined by the Laplacian, depends only on the isomorphism class of the graph (i.e. it is independent of how we label the graph vertexes).
2. Show that L_1 and L_2 are equivalent as integral quadratic forms.
3. Can you make a conjecture regarding when two graphs have equivalent integral quadratic forms? And conversely, if two graphs have equivalent integral quadratic forms, how are the two graphs related to each other?
4. Try to prove your conjectures.