11. Minkowski convex body theorem (01 August 2019).

1. **Blichfeldt's Lemma.** *Consider a set* $A \subset \mathbb{R}^n$ *of volume* $\mathrm{vol}\, A > 1$. *There exist two different points* $x, y \in A$ *such that* $x - y \in \mathbb{Z}^n$.

2. **Minkowski convex body theorem.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded convex* 0*-symmetric set of the volume* $\mathrm{vol}\,\Omega > 2^n$. *Then there exists a non-zero integer point which belongs to* Ω.

3. Mordell's proof of Minkowski Theorem: application of pigeon-hole principle to the set of rational points from

$$\frac{1}{q}\mathbb{Z}^n \cap \Omega.$$

Exercises.

1. Prove statements of Exercises 2 and 3 from Sheet 1 (15July2019) by means of Minkowski theorem.

2. Theorem on linear forms. Consider linear forms

$$L_j(x) = L_j(x_1, ..., x_n) = \sum_{i=1}^{n} \alpha_{i,j} x_i, \quad 1 \le j \le n$$

with determinant $\Delta = \det(\alpha_{i,j})_{1 \le i,j \le n}$. Suppose that positive $\varepsilon_j, 1 \le j \le n$ satisfy

$$\varepsilon_1 \cdots \varepsilon_n \ge |\Delta|.$$

Then the system of inequalities

$$|L_1(x)| \le \varepsilon_1, \qquad |L_j(x)| < \varepsilon_j, \ 2 \le j \le n$$

has a non-zero integer solution $x = (x_1, ..., x_n) \in \mathbb{Z}^n$.

3. Consider linear forms

$$L_j(x) = L_j(x_1, ..., x_m) = \sum_{i=1}^{m} \alpha_{i,j} x_i, \quad 1 \le j \le n$$

Prove that for any $X \ge 1$ there exists $x = (x_1, .., .x_m) \in \mathbb{Z}^m$ such that

$$\max_{1 \le j \le n} ||L_j(x)|| < X^{-\frac{n}{m}}, \quad 1 \le \max_{1 \le i \le m} |x_i| \le X.$$

4. About Diophantine constant. Let $\alpha_1, ..., \alpha_n$ be real numbers
a. Prove that for any $M, t > 0$ the set

$$\Omega(M, T) = \{(x, y_1, ..., y_n) \in \mathbb{R}^{n+1} : \ t^{-n}|x| + nt \max_{1 \le i \le n} |y_i - \alpha_i x| \le M\}$$

is convex, compact, 0-symmetric and has volume

$$\frac{(2M)^{n+1}}{(n+1)n^n}.$$

b. Prove that if $(x, y_1, ..., y_n) \in \Omega(M, T)$ and

$$t^{-n}|x| \neq t \max_{1\leq i\leq n} |y_i - \alpha_i x|,$$

then

$$|x| \left(\max_{1\leq i\leq n} |y_i - \alpha_i x| \right)^n < \left(\frac{M}{n+1} \right)^{n+1}.$$

c. Prove that there exist infinitely many $q \in \mathbb{Z}_+$ with

$$q^{\frac{1}{n}} \max_{1\leq i\leq n} ||\alpha_i q|| \leq \frac{n}{n+1}.$$

5. There exist infinitely many $q \in \mathbb{Z}_+$ such that

$$q \prod_{i=1}^{n} ||\alpha_i q|| < \frac{n!}{(n+1)^n}.$$