13. p-abic numbers (05 August 2019).

1. **Definition 1.** A sequence $\{\xi_n\}_{n=1,2,3,...}$ of points in $\mathbb{Q}$ is called *fundamental* if

$$\forall\,\varepsilon > 0 \;\; \exists\, n_0 \in \mathbb{Z}_+ \;\; \text{such that} \;\; \forall\, m, n \geq n_0 \;\; \text{one has} \;\; |\xi_m - \xi_n| < \varepsilon.$$

2. How to construct $\mathbb{R}$ from $\mathbb{Q}$?
Real numbers are constructed as *classes of equivalence* of sequences $\{\xi_n\}_{n=1,2,3,...}$of rational numbers.
Definition 2. A sequence $\{a_n\}_{n=1,2,3,...}$ converges to zero if

$$\forall\,\varepsilon > 0 \;\; \exists\, n_0 \in \mathbb{Z}_+ \;\; \text{such that} \;\; \forall\, n \geq n_0 \;\; \text{one has} \;\; |\xi_n| < \varepsilon.$$

Definition 3. Two sequences $\{\xi_n\}_{n=1,2,3,...} \subset \mathbb{Q}$ and $\{\xi'_n\}_{n=1,2,3,...} \subset \mathbb{Q}$ are equivalent if the sequence $\{a_n\}_{n=1,2,3,...}$, $a_n = \xi_n - \xi'_n$ converges to zero.

Question. *What is* $|\cdot|$ *here?*

3. p-adic metric on rationals. Fix prime p and let for $m \in \mathbb{Z}$ define

$$\nu_p(m) = \max\{\nu \in \mathbb{Z}_+ : \; p^\nu | m\}.$$

Definition 4. p-adic metric:

$$|\xi|_p = p^{\nu_p(b) - \nu_p(a)} \quad \text{if} \quad \xi = \frac{a}{b} \in \mathbb{Q}, \quad \text{and} \quad |0|_p = 0.$$

4. Properties of p-adic metric.
   1) $|\xi|_p = 0 \iff \xi = 0$;
   2) triangle inequality

$$|\xi + \eta|_p \leq \max(|\xi|_p, |\eta|_p) \leq |\xi|_p + |\eta|_p;$$

   3) $|\xi \cdot \eta|_p = |\xi|_p \cdot |\eta|_p$.

5. **Definition 5.** $f : \mathbb{Q} \to \mathbb{R}$ is a *metric* if
   1) $f(\xi) \geq 0$ and $f(\xi) = 0 \iff \xi = 0$;
   2) triangle inequality

$$f(\xi + \eta) \leq f(\xi) + f(\eta);$$

   3) $f(\xi \cdot \eta) = f(\xi) \cdot f(\eta)$.

6. **Theorem.** (Ostrowski) *The only non-trivial metrics on* $\mathbb{Q}$ *are* $|\cdot|^\alpha$ *where* $0 < \alpha \leq 1$ *and* $|\cdot|_p^\alpha$, *where* p *is a prime and* $\alpha > 0$.

7. What happens in $\mathbb{Q}_p$?
a. All the triangles have two equal edges.
b. If two disks intersect then one one disk is inside another one.
c. Every inner point of a disk is its center.
d. $\sum_{n=1}^{\infty} \xi_n$ converges $\iff$ $\xi_n \to 0, n \to \infty$.