14. Simultaneous approximation to two numbers. (07.08.2019)

For $\alpha_1, \alpha_2 \in \mathbb{R}$ consider irrationality measure function

$$\psi_{\alpha_1,\alpha_2}(t) = \min_{q\in\mathbb{N}:\, q\le t}\ \max_{j=1,2} ||\alpha_j q||.$$

1. For which (α_1, α_2) there exists t_0 such that

$$\psi_{\alpha_1,\alpha_2}(t) = 0$$

for all $t \ge t_0$?

2. Best approximation vectors. Let

$$q_1 < q_2 < ... < q_\nu < q_{\nu+1} < ...$$

be the sequence of points where $\psi_{\alpha_1,\alpha_2}(t)$ is not continuous. Define $a_{a,\nu}, a_{2,\nu}$ by

$$||q_\nu\alpha_1|| = |q_\nu\alpha_1 - a_{1,\nu}|, \quad ||q_\nu\alpha_2|| = |q_\nu\alpha_2 - a_{2,\nu}|$$

Suppose that $a_{1,\nu}, a_{2,\nu}$ are defined uniquely (when it happens?) and consider vectors

$$\boldsymbol{z}_\nu = (q_\nu, a_{1,\nu}, a_{2,\nu}) \in \mathbb{Z}^3, \quad \nu = 1, 2, 3, ...$$

and values

$$\psi_\nu = \max_{j=1,2} ||q_\nu\alpha_j|| = \max_{j=1,2} |q_\nu\alpha_j - a_{j,\nu}|$$

3. For $Q \ge 1$ and $\sigma \le 1/2$ consider parallelepipeds $\Pi(Q, \sigma)$ defined by

$$\Pi(Q,\sigma) = \{\boldsymbol{z} = (x, y_1, y_2) \in \mathbb{R}^3 :\ |x| \le Q,\ \ |y_1 - x\alpha_1| \le \sigma,\ \ |y_2 - x\alpha_2| \le \sigma\}.$$

a. What is its volume?
b. Is it true that the point $\boldsymbol{z}_\nu$ is on the boundary of $\Pi(q_\nu, \psi_\nu)$?
c. Is it true that the point $\boldsymbol{z}_\nu$ is on the boundary of $\Pi(q_{\nu+1}, \psi_\nu)$?
d. What are integer points in $\Pi(q_\nu, \psi_\nu)$ and $\Pi(q_{\nu+1}, \psi_\nu)$?

3. Two successive B.A.
a. Triangle $\mathbf{0}\boldsymbol{z}_\nu\boldsymbol{z}_{\nu+1}$ has no integer points, but vertices.
b. The area of the triangle $\mathbf{0}\boldsymbol{z}_\nu\boldsymbol{z}_{\nu+1}$ is greater than $\frac{q_{\nu+1}\psi_\nu}{100}$ and less than $100q_{\nu+1}\psi_\nu$.

4. Primitivity and independence.
a. g.c.d $(q_\nu, a_{1,\nu}, a_{2,\nu}) = 1$
b. For any ν vectors $\boldsymbol{z}_{\nu-1}$ и $\boldsymbol{z}_\nu$ are independent and can be completed to a basis of $\mathbb{Z}^3$).
c. Suppose that $\alpha_1, \alpha_2, 1$ are independent over $\mathbb{Z}$. Then there exist infinitely many ν such that the vectors $\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}$ are independent.

5. Dipohantine exponents.
a. *Ordinary* Diophantine exponent $\omega(\boldsymbol{\alpha})$ is defined as

$$\omega(\boldsymbol{\alpha}) = \sup\{\gamma \in \mathbb{R} :\ \liminf_{t\to\infty} t^\gamma \psi_{\boldsymbol{\alpha}}(t) < +\infty\}.$$

Prove that $\omega(\boldsymbol{\alpha})$ is supremum over all those γ for which the inequality

$$\max_{j=1,2} ||q\alpha_j|| \leq q^{-\gamma}$$

has infinitely many solutions in $q\mathbb{Z}_+$.
b. *Uniform* Diophantine exponent $\hat{\omega}(\boldsymbol{\alpha})$ is defined as

$$\hat{\omega}(\boldsymbol{\alpha}) = \sup\{\gamma \in \mathbb{R} : \limsup_{t\to\infty} t^{\gamma}\psi_{\boldsymbol{\alpha}}(t) < +\infty\}.$$

Prove that $\hat{\omega}(\boldsymbol{\alpha})$ is supremum over all those γ for which there exists Q_0 such that the system

$$\begin{cases} \max_{j=1,2} ||q\alpha_j|| \leq Q^{-\gamma} \\ q \leq Q \end{cases}$$

is solvable for every $Q \geq Q_0$ имеет натуральное решение q.
c. Trivial inequality $\hat{\omega}(\boldsymbol{\alpha}) \leq \omega(\boldsymbol{\alpha})$.

6. Bounds for uniform exponent: $\hat{\omega}(\boldsymbol{\alpha}) \in \left[\frac{1}{2}, 1\right]$, provided $(\alpha_1, \alpha_2) \notin \mathbb{Q}^2$.

7. Main property. For any $\gamma < \hat{\omega}(\boldsymbol{\alpha})$ for the B.A. vectors to $\boldsymbol{\alpha}$ for ν large enough one has

$$\psi_\nu \leq q_{\nu+1}^{-\gamma}.$$

8. Suppose that $\gamma < \hat{\omega}$. Suppose that the vectors $\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}$ are independent. Then for large ν one has

$$q_{\nu+1} \geq \frac{q_\nu^{\frac{\gamma}{1-\gamma}}}{100}.$$

Suggestion: consider the determinant

$$0 \neq \begin{vmatrix} q_{\nu-1} & a_{1,\nu-1} & a_{2,\nu-1} \\ q_\nu & a_{1,\nu} & a_{2,\nu} \\ q_{\nu+1} & a_{1,\nu+1} & a_{2,\nu+1} \end{vmatrix} = \begin{vmatrix} q_{\nu-1} & a_{1,\nu-1} - \alpha_1 q_{\nu-1} & a_{2,\nu-1} - \alpha_2 q_{\nu-1} \\ q_\nu & a_{1,\nu} - \alpha_1 q_\nu & a_{2,\nu} - \alpha_2 q_\nu \\ q_{\nu+1} & a_{1,\nu+1} - \alpha_1 q_{\nu+1} & a_{2,\nu+1} - \alpha_2 q_{\nu+1} \end{vmatrix}.$$

9. **Theorem.** (V. Jarnik) *Suppose that $\alpha_1, \alpha_2, 1$ are linear independent over $\mathbb{Z}$. Then*

$$\omega(\boldsymbol{\alpha}) \geq \hat{\omega}(\boldsymbol{\alpha}) \cdot \frac{\hat{\omega}(\boldsymbol{\alpha})}{1 - \hat{\omega}(\boldsymbol{\alpha})}.$$