5. Normal numbers (22 July 2019).

1. **Definition 1.** Number α is normal with respect to the base $q > 1, q \in \mathbb{Z}_+$ if fractional parts $\{\alpha q^n\}, n = 1, 2, 3, ...$ are U.D.
Borel proved that almost all (in the sense of Lebesgue measure) numbers in $\mathbb{R}$ are normal to any base. But it is not even known if $\sqrt{2}, e$, or π are normal with respect to base 2.

2. A finite sequence of simbols $\delta_1\delta_2...\delta_n$, $\delta_j \in \{0, 1\}, n = n(t)$ is called t-normal sequence (with respect to base 2) if every combination of t symbols $\eta_1...\eta_t$ ($\eta_j \in \{0, 1\}$) occurs as a subword in $\delta_1\delta_2...\delta_n$ and occurs only once. What is $n(t)$?

3. System ρ_t. Write the sequence $\underbrace{11...1}_{t}$. and then step by step write a symbol to the right of the sequence, following the rule "0 is better than 1". You should stop when you cannot continue without repetition. Denote the sequence you obtain as ρ_t. What is at the end of this sequence?
Theorem 1. *ρ_t is a t-normal sequence.*

4. Let $\psi(k) \to +\infty, k \to \infty$ and ρ'_k be the system of symbols ρ_k without last $k-1$ symbols. Consider the number

$$\kappa = 0.\underbrace{\rho'_1\rho'_1...\rho'_1}_{\psi(1)}\underbrace{\rho'_2\rho'_2...\rho'_2}_{\psi(2)}\ ...\ \underbrace{\rho'_k\rho'_k...\rho'_k}_{\psi(k)}\ ....$$

Theorem 2. (N. Korobov) *Number κ is normal in the base 2.*

5. t-normal sequences are complete cycles in de Bruijn graphs.
a. What are de Bruijn graphs?
b. **Definition.** 2-graph is a finite oriented graph such that just two edges come to each vertex and just two edges go out of each vertex.
c. What is the *doubling* G^* of 2-graph G?
d. **de Bruijn's theorem.** *Let G be a 2-graph with m points and has exactly M complete cycles, then G^* has exactly 2^mM complete cycles.*

Exercises.

1. Prove that a number α is normal with respect to the base 2 if and only if for its dyadic espansion

$$\alpha = 0.\delta_1\delta_2...\delta_\nu...$$

for each combination of symbols $\eta_1..\eta_t$ for the number

$$N_q(\eta_1..\eta_t) = |\{j \le q : \ \delta_j\delta_{j+1}...\delta_{j+t-1} = \eta_1..\eta_t\}|$$

the asymptotic equality

$$N_q(\eta_1..\eta_t) = \frac{q}{2^t} + o(q), \ \ q \to \infty$$

holds.

2. Finish the proof of Theorem 1.

3. Prove that there exists α such that for the discrepancy D_q of the sequence $\{\alpha 2^n\}$ one has $D_q = O(\sqrt{q})$. (Suggestion: choose $\psi(k)$ optimally.)

4. Prove that for any α for the discrepancy D_q of the sequence $\{\alpha 2^n\}$ one has

$$\limsup_{q\to\infty} \frac{D_q}{\log q} > 0.$$

5. Sums of fractional parts.
a. Prove that there exists α such that

$$\sum_{k=1}^{q} \{\alpha 2^k\} = \frac{q}{2} + o(q), \quad q \to \infty.$$

b. Prove that for arbitrary function $\varphi(q)$ such that $\lim_{q\to\infty} \varphi(q) = \infty$ there exists α such that

$$\sum_{k=1}^{q} \{\alpha 2^k\} = \frac{q}{2} + O\left(\varphi(q)\right), \quad q \to \infty.$$

6. Finish the proof of **de Bruijn's theorem** and deduce the following
Corollary. For each positive integer n, there are exactly $2^{2^{n-1}-n}$ complete cycles of length $N = 2^n$. (See M. Hall, *Combinatorial Theory*, Second Edition 1983 John Wiley & Sons, Inc, Chapter 9).