6. From Basic Number Theory (23 July 2019).

First of all I would like to recall the definitions of
a. Euler Function

$$\varphi(n) = |\{x \in \mathbb{Z}:\ 1 \le x \le n,\ (x,n) = 1\}|;$$

b. Möbius function

$$\mu(n) = \begin{cases} 1, & n = 1, \\ 0, & p \text{ prime},\ p^2|n, \\ (-1)^r, & n = p_1 \cdots p_r \ - \text{ different primes.} \end{cases}$$

These functions are multiplicative. A function $f(x), x \in \mathbb{Z}_+$ is *multiplicative* if

$$f(nm) = f(n)f(m) \text{ for all } \ n, m \in \mathbb{Z}_+ \text{ under the condition } (n,m) = 1.$$

1. Well known formulas:

$$\sum_{d:\, d|n} \mu(d) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases} \qquad \varphi(n) = n \cdot \sum_{d:\, d|n} \frac{\mu(d)}{d}, \qquad \sum_{d:\, d|n} \varphi(d) = n.$$

2. Möbius inversion formula. Let $g(x)$ and $G(x)$ be two functions of $x \in \mathbb{Z}_+$. Then

$$G(n) = \sum_{d|n} g(n) \iff g(n) = \sum_{d|n} \mu(d) G\left(\frac{n}{d}\right).$$

3. What are sums $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$?

4. Consider the sum

$$S_x(Q) = \sum_{q:\, q \le Q} \ \sum_{1 \le a \le xq} \ \sum_{d:\, d|\text{н.о.д.}(a,q)} \mu(d)$$

Prove that

$$S_1(Q) = \sum_{q \le Q} \varphi(q).$$

What is the meaning of this sum when $x \in [0,1]$?

5. Find limits

$$\lim_{Q\to\infty} \frac{S_1(Q)}{Q^2}, \quad \lim_{Q\to\infty} \frac{S_x(Q)}{Q^2} \text{ and } \lim_{Q\to\infty} \frac{S_x(Q)}{S_1(Q)}.$$

6. Consider *Farey series*

$$\mathcal{F}_Q = \left\{ \frac{p}{q} \in [0,1]: \quad q \le Q \right\}.$$

a. Now many numbers are there in $\mathcal{F}_Q$?
b. Calculate the limit

$$\lim_{Q\to\infty} \frac{|\{\xi \in \mathcal{F}_Q:\ \xi \le x\}|}{|\mathcal{F}_Q|}, \quad x \in [0,1].$$

7. Consider Stern-Brocot sequences F_n. What is the limit

$$\lim_{n\to\infty} \frac{|\{\xi \in F_n :\ \xi \le x\}|}{|F_n|}?$$

Exercises.

1. For two functions $f(n), g(n), n \in \mathbb{Z}_+$ define the operation

$$(f \circ g)(n) = \sum_{d|n} f(d)g(n/d).$$

This operation is called *Dirichlet convolution.* Prove that it is commutative (that is $f\circ g = g\circ f$) and associatove (that is $f \circ (g \circ h) = (f \circ g) \circ h$).

2. Write formulas from 1. and 2. in terms of $\circ$.
Suggestion: you should use functions

$$I(x) = 1\ \forall x,\ \ E(x) = x\ \forall\, x,\ \ J(x) = \begin{cases} 1, & x = 1, \\ 0, & x > 1. \end{cases}$$

3. Let $\tau(n) = \sum_{d|n} 1$ be the number of divisors of n and $\sigma(n) = \sum_{d|n} d$ be the sum of divisors of n. Calculate sums
a. $\sum_{d|n} \mu(d)\tau(n/d)$;
b. $\sum_{d|n} \mu(d)\sigma(n/d)$.
You will obtain certain formulas. Write them in terms of $\circ$.

4. Dirichlet series. Consider infinite sums

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^2} \text{ and } \sum_{m=1}^{\infty} \frac{g(m)}{m^2}.$$

Let us write the product in a form

$$\left(\sum_{n=1}^{\infty} \frac{f(n)}{n^2}\right) \cdot \left(\sum_{m=1}^{\infty} \frac{g(m)}{m^2}\right) = \sum_{k=1}^{\infty} \frac{h(k)}{k^2}.$$

What is $h(\cdot)$? (We suppose that we can change the order of INFINITE summation everywhere, for example this is possible when the series are absolutely convergent.)

5. Find all positive integers n, x satisfying $\varphi(nx) = \varphi(x)$.

6. Calculate the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \tau(n)}{n \log n}.$$