9. Irrationality measure functions (29 July 2019).

1. For a real α consider the*ordinary* irrationality measure function

$$\psi_\alpha(t) = \min_{1\le x\le t,\, x\in\mathbb{Z}} ||x\alpha||,$$

Lagrange constant

$$\lambda(\alpha) = \liminf_{t\to+\infty} t\cdot\psi_\alpha(t),$$

and Dirichlet constant

$$d(\alpha) = \limsup_{t\to+\infty} t\cdot\psi_\alpha(t).$$

What is the maximal and minimal possible values of value of $\lambda(\alpha)$ and $d(\alpha)$?

The Lagrange spectrum $\mathbb{L}$ is defined as

$$\mathbb{L} = \{\lambda\in\mathbb{R}:\ \text{there exists } \alpha\in\mathbb{R} \text{ such that } \lambda=\lambda(\alpha)\}.$$

The Dirichlet spectrum $\mathbb{D}$ is defined as

$$\mathbb{D} = \{d\in\mathbb{R}:\ \text{there exists } \alpha\in\mathbb{R}\ \text{ such that } d=d(\alpha)\}.$$

Define

$$\xi_\nu = ||q_\nu\alpha|| = |q_\nu\alpha - p_\nu|.$$

2. Minkowski function. Recall the Legendre theorem on continued fractions. This theorem says that if

$$\left|\alpha - \frac{A}{Q}\right| < \frac{1}{2Q^2},\ \ (A,Q)=1 \tag{1}$$

then the fraction $\frac{A}{Q}$ is a convergent fraction for the continued fraction expansion of α. The converse statement is not true. It may happen that $\frac{A}{Q}$ is a convergent to α but (1) is not valid. We consider the sequence of the denominators of the convergents to α for which (1) is true. Let this sequence be

$$Q_0 < Q_1 < \cdots < Q_n < Q_{n+1} < \cdots .$$

Then for $\alpha\notin\mathbb{Q}$ the function $\mu_\alpha(t)$ is defined by

$$\mu_\alpha(t) = \frac{Q_{n+1}-t}{Q_{n+1}-Q_n}\cdot||Q_n\alpha|| + \frac{t-Q_n}{Q_{n+1}-Q_n}\cdot||Q_{n+1}\alpha||,\quad Q_n\le t\le Q_{n+1}.$$

Note that for every ν one of the consecutive convergent fractions $\frac{p_\nu}{q_\nu}$, $\frac{p_{\nu+1}}{q_{\nu+1}}$ to α satisfies (1). So either

$$(Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1})$$

for some ν and

$$||Q_n\alpha|| = \xi_\nu,\ \ ||Q_{n+1}\alpha|| = \xi_{\nu+1},$$

or

$$(Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1})$$

for some ν and

$$||Q_n\alpha|| = \xi_{\nu-1}, \ \ ||Q_{n+1}\alpha|| = \xi_{\nu+1},$$

How this function is related to the Minkowski theorem form Sheet 8 (No. 7)?

3. Define

$$\mathfrak{m}(\alpha) = \limsup_{t\to+\infty} t \cdot \mu_\alpha(t).$$

What is the minimal and the maximal value of $\mathfrak{m}(\alpha)$?

Exercises.

1. Prove that

a.

$$q_\nu\xi_\nu = \frac{1}{\alpha_\nu^* + \alpha_{\nu+1}} = \frac{1}{(\alpha_{\nu+1}^*)^{-1} + (\alpha_{\nu+2})^{-1}} = \frac{\alpha_{\nu+1}^*\alpha_{\nu+2}}{\alpha_{\nu+1}^* + \alpha_{\nu+2}};$$

b. $\xi_\nu/\xi_{\nu+1} = \alpha_{\nu+2};$

c. $\xi_{\nu+1} = \xi_{\nu-1} - a_{\nu+1}\xi_\nu;$

2. Prove that $\psi_\alpha(t) \leq 1/t$ for all t.

3. Find $\lambda(\alpha_N)$ and $d(\alpha_N)$, where $\alpha_N = [\overline{N}]$.

4. Prove that there exist a sequence $t_\nu \to \infty$, such that

$$\psi_{\sqrt{2}}(t_\nu) > \psi_{(\sqrt{5}+1)/2}(t_\nu),$$

and that there exist a sequence $r_\nu \to \infty$, such that

$$\psi_{\sqrt{2}}(r_\nu) < \psi_{(\sqrt{5}+1)/2}(r_\nu),$$

5. What is $\liminf_{t\to+\infty} t \cdot \mu_\alpha(t)$?

6. Find $\mathfrak{m}(\alpha)$ for $\alpha = \sqrt{2}$ and $\alpha = \frac{1+\sqrt{3}}{2} = [1; 2, 1, 2, 1, 2, 1, ...]$.

7. Prove that

$$\mu_{\sqrt{2}}(t_\nu) < \mu_{(\sqrt{5}+1)/2}(t_\nu), \ \ \forall t \geq 1.$$