ANALYSIS AND TOPOLOGY HOMEWORK 1 (07/17)

(1) Show that when $r \neq 1$
$$\sum_{k=1}^{n} r^k = \frac{1-r^{n+1}}{1-r}.$$

(2) Show that the partial sum S_n of $\sum_{k=0}^{\infty} \frac{1}{k!}$ satisfies
$$e - S_n < \frac{1}{n!n}.$$

(3) Prove that e^x is an increasing function.

(4) Define
$$t_n = \left(1 + \frac{1}{n}\right)^n, \qquad s_n = \sum_{k=0}^{n} \frac{1}{k!}$$
Suppose $\lim_{n\to\infty} t_n$ exists and equals to t.

(a). Show that $t_n \leq s_n$, and then $t \leq e$;

(b). Show that if $n \geq m$
$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$
Then use it to show $e \leq t$.

Therefore (1) and (2) together proves
$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$
(This is the sequence definition of e.)

(5) Find the derivative $\frac{d}{dx}e^{cx}$ where c is a constant (Do not use the chain rule).