# ANALYSIS AND TOPOLOGY HOMEWORK 2 (07/18)

(1) Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for any integer $n$. (Hint: Use the definition of $e^x$.)

(2) Review of integration by parts (IBP):

Let $f(x)$ and $g(x)$ be two functions. Recall that the product rule in differentiation is

$$[fg]' = f'g + fg'.$$

We integrate both sides of the above equation to get

$$fg = \int f'g\ dx + \int fg'\ dx.$$

Move a term to the other side, we get the **integration by parts** formula

$$\int f'g\ dx = fg - \int fg'\ dx.$$

(a). For example, compute $\int xe^x\ dx$ by setting $f'(x) = e^x$ and $g(x) = x$ in above formula. What happens if you choose $f'(x) = x$ and $g(x) = e^x$?

(b). Compute $\int x \cos x\ dx$.

(c). Compute $\int x^2 e^x\ dx$.

(3) Review of the Chain rule:

The composition of two functions $f(x)$ and $g(y)$ is

$$g \circ f(x) = g(f(x))$$

For example, the composition of $g(x) = x^2$ and $f(x) = e^x$ is $g \circ f(x) = (e^x)^2 = e^{2x}$. To find the derivative of $g \circ f$ we assume both $f$ and $g$ are differentiable. Then consider the limit

$$\lim_{h\to 0} \frac{g \circ f(x+h) - g \circ f(x)}{h} = \lim_{h\to 0} \frac{g(f(x+h)) - g(f(x))}{h}$$

We write

$$g(f(x+h)) = g(f(x) + f(x+h) - f(x))$$

and denote $s = f(x+h) - f(x)$. Then we have

$$\frac{g(f(x+h)) - g(f(x))}{h} = \frac{g(f(x)+s) - g(f(x))}{s} \frac{f(x+h) - f(x)}{h}$$

Note that as $h \to 0$, $s = f(x+h) - f(x) \to 0$ since $f$ is continuous (it's differentiable hence continuous). Therefore

$$\lim_{h\to 0} \frac{g(f(x+h)) - g(f(x))}{h} = \left(\lim_{s\to 0} \frac{g(f(x)+s) - g(f(x))}{s}\right) \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right).$$

The first limit on the right hand side gives $g'(f(x))$ and the second limit gives $g'(x)$. Then we obtain the chain rule

$$[g(f(x))]' = [g'(f(x))]f'(x).$$

(a). Find the derivative of $\cos(e^x)$ and $e^{\cos x}$.

(4) The inverse function of $f(x) = e^x$.
The inverse function $y = f^{-1}(x)$ of any function $f$ is such that

$$f \circ f^{-1}(x) = f^{-1} \circ f(x) = x.$$

It can be obtained by solving for $y$ in $x = f(y)$. For example, $f(x) = x^3$ has the inverse function $y = f^{-1}(x) = x^{1/3}$ by solving $x = y^3$.

(a). Verify $f \circ f^{-1}(x) = f^{-1} \circ f(x) = x$ for $f(x) = x^3$.

Note that the inverse function of $f(x)$ exists only when $f$ is one-to-one on its domain. For example, $f(x) = x^2$ doesn't have an inverse function on the whole $(-\infty, \infty)$.

(b). The exponential function $f(x) = e^x$ has an inverse function on $(-\infty, \infty)$. Why?

(c). We denote the inverse function of $e^x$ by $\ln x$. What is its domain? Draw its graph.

(d). Use the chain rule above to prove $(\ln x)' = 1/x$ on its domain.

(5) Find all the eigenfunctions of the differential operator $\left(\frac{d}{dx}\right)^2 - 2\frac{d}{dx}$ corresponding to the eigenvalue $p$.