Projects by Moshchevitin.

Research project 1.
Distribution of quickly growing sequences.

Problem 1. Construct α such that the sequence $\{\alpha n!\}$ is U.D.

Problem 2. Construct α such that the sequence $\{\alpha n!\}$ is not dense in $[0, 1]$.

Problem 3. To give an (alternative) proof of Aleksenko's Theorem: There exists α such that the sequence $\{\alpha n!\}$ has discrepancy $D_q = O(\log q)$,

Problem 4. Is it true that the sequence $\{\alpha n!\}$ may have "any" intermediate order of discrepancy? (Suggested by D. Bilyk some time ago, unsolved and easy.)

Problem 5. Construct α such that the sequence $\{\alpha 2^n\}$ is U.D.

Problem 6. Construct α such that the sequence $\{\alpha 2^n\}$ is not dense in $[0, 1]$.

Problem 7. To understand the proof of Levin's Theorem: There exists α such that the sequence $\{\alpha 2^n\}$ has discrepancy $D_q = O(\log^2 q)$,

Problem 8. Is it true that the sequence $\{\alpha 2^n\}$ may have order of discrepancy $D_q = O(\log q)$? (Unsolved and difficult.)

Problem 9. Can you say something (new) about sequences $\{\alpha f(n)\}$ where $f(n)$ has intermediate growth between 2^n and $n!$?

Research project 2.
Markoff and Dirichlet spectra compared.

For irrational α we consider its Lagrange constant

$$\lambda(\alpha) = \liminf_{n\to\infty} q_n||q_n\alpha||$$

and Dirichlet constant

$$d(\alpha) = \limsup_{n\to\infty} q_{n+1}||q_n\alpha||.$$

Lagrange spectra $\mathbb{L}$ is the set of all values of $\lambda(\alpha)$ and Dirichlet spectra $\mathbb{D}$ is the set of all values of $d(\alpha)$.

Problem 1. Find the maximal element in $\mathbb{L}$ and the minimal element in $\mathbb{D}$.

Problem 2. Prove that the points from Problem 1 are isolated.

Problem 3. To give an (alternative) proof of Hall's Theorem: There exists λ^* such that $[0, \lambda^*] \in \mathbb{L}$.

Problem 4. To give an (alternative) proof of Diviš-Nowak Theorem: There exists d^* such that $[d^*, 1] \in \mathbb{D}$.

Problem 5. Prove that there exists There exists (λ^{**}, d^{**}) such that for any $(\lambda, d) \in [0, \lambda^{**}] \times [d^{**}, 1]$ there exists α with $\lambda(\alpha) = \lambda, d(\alpha) = d$. (Unsolved and easy.)

Problem 6. To understand Markoff and Diviš theorems about the structure of the discrete parts of Lagrange and Dirichlet spectra.

Problem 7. To find all (λ, d) there exists α with $\lambda(\alpha) = \lambda, d(\alpha) = d$. (Unsolved and in general hopeless.)

Problem 8. To give partial solutions of Problem 7.

Research project 3.
Minkowski question mark function $?(x)$.

Minkowski question mark fucntion is defined as

$$?(x) = \lim_{n\to\infty} \frac{|F_n \cap [0,x]|}{|F_n|}, \quad x \in [0,1]$$

Problem 1. Prove that for $x \in \mathbb{Q}$ the value $?(x)$ is a dyadic rational number and for quadratic irrationality x we have $?(x) \in \mathbb{Q}$.

Problem 2. It is known that if $?'(x)$ exists then either $?'(x) = 0$ or $?'(x) = +\infty$. We will call $A \in [0,1]$ a *possible derivative* of $?(x)$ at point x if there exists a sequence of $y_j \to x$ such that

$$\lim_{j\to\infty} \frac{?(y_j) - ?(x)}{y_j - x} = A.$$

Find a good lover bound for the quantity

$$\omega = \inf_{x:?'(x)\neq 0,+\infty} \frac{A_1}{A_2},$$

where A_i are possible derivatives of $?(x)$ at the point x. (Unsolved and easy.) Find the exact value of ω. (Unsolved.)

Problem 3. Find all rational stable points of the function $?(x)$, that is the solutions of equation $?(x) = x$.

Problem 4. Find all stable points of the function $?(x)$, that is the solutions of equation $?(x) = x$. (Unsolved and probably hopeless.)

Problem 5. Prove that a solution of $?(x) = x$ cannot be a Liouville number.

Problem 6. Prove that

$$\left| \sum_{j=1}^{2^n} \left(\xi_{j,n} - \frac{j}{2^n} \right)^2 - 2^n \int_0^1 (?(x) - x)^2 dx \right| \leq 4,$$

where $\xi_{j,n}$ are elements of F_n.

Problem 7. Improve bound 4 from the previous problem. (Unsolved.)

Problem 8. Prove or disprove that that

$$\sum_{j=1}^{2^n} \left(\xi_{j,n} - \frac{j}{2^n} \right)^2 - 2^n \int_0^1 (?(x) - x)^2 dx \to 0, \ \ n \to \infty.$$

(Unsolved.)

Research project 4.

Best Diophantine approximation in higher dimension.

Let $\boldsymbol{z}_\nu = (q_\nu, a_{1,\nu}, ..., a_{n,\nu}) \in \mathbb{Z}^{n+1}$ be vectors of the *best simultaneous approximation* to the numbers $\alpha_1, ..., \alpha_n$ that is the vectors defined by

$$\min_{q \le q_\nu} \max_{1 \le j \le n} ||q\alpha_j|| = \max_{1 \le j \le n} |q_\nu \alpha_j - a_{j,\nu}|$$

and

$$\omega = \sup\{\gamma : \liminf_{n\to\infty} q_\nu^\gamma \max_{1 \le j \le n} ||q_\nu \alpha_j|| < \infty\},$$

$$\hat{\omega} = \sup\{\gamma : \limsup_{n\to\infty} q_{\nu+1}^\gamma \max_{1 \le j \le n} ||q_\nu \alpha_j|| < \infty\}$$

be the *ordinary* and the *uniform* Diophantine exponents.

Of course all these objects can be defined in terms of "multidimensional" irrationality measure functions.

Problem 1. Prove an analog of Valen's theorem for two consecutive best approximations.

Problem 2. Proof that if $1, \alpha_1, ..., \alpha_n, n \ge 2$ are linearly independent over $\mathbb{Q}$ then there exist infinitely many ν such that $\boldsymbol{z}_{\nu-1}, \boldsymbol{z}_\nu, \boldsymbol{z}_{\nu+1}$ are independent in $\mathbb{R}^{n+1}$.

Problem 3. How the result of Problem 1 cam be generalized to several (more than three) consecutive vectors of best approximation? This problem has many aspects. It is related to multidimensional geometry and multidimensional continued fractions.

Problem 4. Give an (alternative) proof of **Jarnik's theorem**: if $1, \alpha_1, \alpha_2$, are linearly independent over $\mathbb{Q}$ then

$$\omega \ge \hat{\omega} \cdot \frac{\hat{\omega}}{1-\hat{\omega}}.$$

Problem 5. Give an (alternative) proof of the statement if $1, \alpha_1, \alpha_2, \alpha_2$ are linearly independent over $\mathbb{Q}$ then

$$\omega \ge \hat{\omega} \cdot \left(\frac{\hat{\omega}}{1-\hat{\omega}} + \sqrt{\left(\frac{\hat{\omega}}{1-\hat{\omega}}\right)^2 + \frac{4\hat{\omega}}{1-\hat{\omega}}} \right).$$

Problem 6. Give a version of the result from Problem 4 in terms of functions of approximation. (Unsolved and easy.)

Problem 7. Prove optimality of the results from problems 3, 4, 5.