LINEAR ALGEBRA HOMEWORK 5 – FRIDAY 8/4

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Exercise 0.1. *Prove that for $A \in M_{n,n}$*

$$\det A^T = \det A.$$

Thus it doesn't matter whether we think of $\det$ *as a function of n row vectors or n column vectors.*
(Hint: Clearly understand the cases $n = 2, 3$ first. Observe that $S_3 \to S_3$, $\sigma \mapsto \sigma^{-1}$, is a bijection.)

Exercise 0.2. *Let $x, y \in M_{n,n}$. Recall that x, y are translates of each other iff there exists an invertible matrix g such that $y = g^{-1}xg$. Prove your assertions.*
(a) Suppose $\det x \neq \det y$. Can x, y be translates of each other, i.e. can $[x] = [y]$?
(b) Suppose $\det x = \det y$. Does this imply that $[x] = [y]$?

Exercise 0.3. WRITE UP *Let $x_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and let $[x_0]$ be its translation class in $M_{2,2}$. Show that there is a surjective map*

$$\pi : [x_0] \to \mathbb{P}^1 := \text{the set of all lines in } F^2$$

given by $x \mapsto \ker x$. Here, a line in F^2 is a one dimensional subspace of F^2. Can you describe the set $\pi^{-1}(\ker x)$ for each x? Prove your assertions.

Exercise 0.4. *Given that Δ and σ are respectively a hull and a beam in $\mathbb{R}^n$, verify that $\Delta^\vee$ and $\sigma^\vee$ are respectively a hull and a beam in $\mathbb{R}^n$. Moreover, we have*

$$(\Delta^\vee)^\vee = \Delta, \quad (\sigma^\vee)^\vee = \sigma.$$

Exercise 0.5. WRITE UP *Verify that if Δ is a perfect hull, then $\Delta^\vee$ is also perfect. This shows that perfect hulls come in pairs!*

Exercise 0.6. *If Δ is a perfect hull, prove that the interior Δ° of Δ, i.e. the subset of points in Δ not on its bounding hyperplanes, contains exactly one integral point, namely the origin 0.*

Exercise 0.7. *Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2,2}$, and let t be a variable. Write down the polynomial function $\det(A - tI) \in F[t]$. What is its degree? What is the coefficient of the highest power of t and the lowest power of t for this polynomial? Generalize your answers to $n \times n$ matrices.*

Exercise 0.8. WRITE UP *Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_{2,2}(\mathbb{C})$. Find an invertible matrix B such that $B^{-1}AB$ is upper triangular.*