LINEAR ALGEBRA HOMEWORK 6 – DUE TUESDAY 8/8

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Throughout this homework, take $F = \mathbb{C}$, $x \in \operatorname{End} F^n \equiv M_{n,n}(F)$, and $V = F^n$. You should try to write clean and short proof as much as possible. The level of clarity demonstrates your level of true undertanding of the problem.

Exercise 0.1. WRITE UP *Let Δ_1, Δ_2 be hulls in $\mathbb{R}^n$ containing 0. Show that if*

$$\Delta_1 \subset \Delta_2 \Longrightarrow \Delta_1^\vee \supset \Delta_2^\vee.$$

Conclude that if a hull Δ in $\mathbb{R}^n$ contains a **ball**, *i.e. $\{x \in \mathbb{R}^n | x \cdot x \leq \lambda\}$ for some $\lambda > 0$, then $\Delta^\vee$ must be contained in a ball in $\mathbb{R}^n$.*

Definition 0.2. *Let $x \in \operatorname{End} V$. We call $\lambda \in F$ a* **root** *of $p_x(t)$ (or simply a root) if*

$$p_x(\lambda) = \det(x - \lambda I) = 0.$$

We call x a **potent** *map if $(x - \lambda I)^k = 0$ for some $k > 0$ and some $\lambda \in F$. In this case, we call λ a potent root of x.*

Recall that x has a **diagonal form** (DF) if x can be translated to a diagonal matrix, i.e. $g^{-1}xg$ is diagonal for some invertible g. Equivalently, there is a basis (v) of V such that the matrix $m(x)$ in this basis is diagonal.

Exercise 0.3. *Let $x \in \operatorname{End} V$ be a potent map. Prove that*
(a) if $\lambda = 0$ is a potent root of x then $I + x$ is invertible.
(Hint: If x were a number, what can you say about $\frac{1}{1+x}$ from calculus?)
(b) x has exactly one potent root λ. Hence we say that x is λ-potent.
(c) x is invertible iff its potent root is nonzero.
(d) Any positive power x^p is potent.
(e) If x is ADF, then x is a multiple of I_n.
(f) If x is 1-potent and if x is ADF, then $x = I_n$.

More generally for $x \in \operatorname{End} V$, we say that $v \in V$ is a **λ-potent vector** of x, if for some $k > 0$

$$(x - \lambda I)^k v = 0.$$

Put

$$V_\lambda := \{v \in V |\ v \text{ is } \lambda\text{-potent vector of } x\}.$$

We call this **the λ-potent space** of x. If $V_\lambda \neq (0)$, we call λ a potent root of x. Note that x is λ-potent map iff $V = V_\lambda$. Beware that we use the term 'potent' for maps, vectors, and spaces, and each use is different though they are all closely related.

Exercise 0.4. WRITE UP *Clearly, $V_\lambda \supset \ker(x - \lambda I)$. If x is ADF, then how is each V_λ related to $\ker(x - \lambda I)$? Is the converse of your assertion true? Prove all your assertions.*

Exercise 0.5. *Let* $x = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$. *Find a basis* (v) *of* $V = F^2$ *such that the matrix* $(m(x)_{ij})$ *of* x *in this basis is UTF (upper triangular form).*