2017 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

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0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). For simplicity, you can think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, ...$ Likewise a linear map means an F-linear, and a matrix means a matrix with entries in F. Put

$$\begin{aligned}
\mathrm{Hom}(U,V) &:= \{\text{linear maps } U \to V\} \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \quad \text{the algebra of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V | f \text{ is bijective}\} \\
\mathrm{Aut}\,_n F &:= \mathrm{Aut}\, F^n \\
\mathrm{Aut}\,_n \mathbb{Z} &= \mathrm{Aut}\, \mathbb{Z}^n, \quad \text{the set of automorphisms } \mathbb{Z}^n \to \mathbb{Z}^n \\
M_n \equiv M_{n,n}(F) &:= \text{the algebra of } n \times n \text{ matrices} \\
I &\equiv I_n := [e_1, .., e_n], \quad \text{the identity matrix in } M_n.
\end{aligned}$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be quite thoroughly studied in class during the first two weeks.

1. Project 1: From subspaces to linear systems

If $\mathcal{L}$ is homogeneous linear system of m equations in n variables $x = (x_1, .., x_n)$ over a field F, we know that its solution set $sol(\mathcal{L})$ is an F-subspace of F^n. This project is about the converse problem.

Throughout this project, let V, W be given F-subspaces of F^n.

Problem 1.1. *Is there a linear system $\mathcal{L}$ such that*

$$V = sol(\mathcal{L}) \ \ ?$$

Problem 1.2. *When such a system exists, can you describe* all *such linear systems $\mathcal{L}$?*

Problem 1.3. *Among those you found in Problem 1.2, can you describe those with the minimum number of equations?*

Problem 1.4. *Give a simple algorithm – as simple as you can think of – to generate such a minimal system. What information about V do you need as input to your algorithm?*

Problem 1.5. *Give a simple algorithm to compute a basis of the F-subspace $V \cap W$. What information about V and W do you need as input to your algorithm?*

As always, it is a good idea to experiment with simple examples. In the end, try to make your solutions to these problems as simple and as easy to understand as possible.

2. Project 2: Solving Polynomial Matrix Equations

We are quite familiar with the problem of solving a polynomial equation in one variable like

$$a_0 + a_1 x + \cdots + a_n x^n = 0$$

in a *field* F. In our first research project, we will explore the problem of solving certain polynomial equations in one variable, but in the *algebra* M_n. As a warm up, let's begin with

Problem 2.1. *Take the complex numbers $F = \mathbb{C}$. For each $n = 2, 3, 4, ...$, solve the matrix equation*

$$\boxed{x^2 = I_n}$$

in the algebra M_n. What if we replace $\mathbb{C}$ by the real numbers $F = \mathbb{R}$?

That is to say: *describe* the set of solutions. Two obvious solutions are $x = \pm I_n$. To find some (or all) other solutions, let us consider the "*symmetry*" of this problem. Notice that if x is a solution and $g \in \mathrm{Aut}_n$, then gxg^{-1} (also called a **translate** of x) is also a solution. Since Aut_n is infinite, they are potentially infinitely many solutions to the equation $x^2 = I$. Thus, we ask: *can we produce a list of all solutions, no two of which are translate of each other? Can we describe such a list?* Observe that already for $n = 2$, if we write $x = (x_{ij})$, then the equation

$$x^2 = I_2$$

is equivalent to a system of 4 quadratic equations in 4 variables

$$\begin{aligned} x_{11}x_{11} + x_{12}x_{21} &= 1 \\ x_{11}x_{12} + x_{12}x_{22} &= 0 \\ x_{21}x_{11} + x_{22}x_{21} &= 0 \\ x_{21}x_{12} + x_{22}x_{22} &= 1. \end{aligned}$$

Problem 2.2. *More generally, still taking $F = \mathbb{C}$ or $\mathbb{R}$, for each $k, n = 2, 3, 4, ...$, solve the matrix equation*

$$\boxed{x^k = I_n}$$

in the algebra M_n.

Problem 2.3. *Do the same for*

$$\boxed{x^k = x.}$$

Problem 2.4. *Do the same for*

$$\boxed{x^k = 0.}$$

3. Project 3: From Linear Algebra to Combinatorial Geometry

In this project, *we shall let $F = \mathbb{R}$ and fix a positive integer n.* The project will be about enumerating certain special objects defined using the *linear and geometric* structure of $\mathbb{R}^n$, but also satisfying certain 'integral' properties at the same time. Let's begin with some notations and definitions, which we will go over in class in more details during the first two weeks. Put

$$\mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}.$$

Given two points $x, y \in \mathbb{R}^n$, the **edge** (or line segment) connecting x, y is the set

$$\overline{xy} := \{(1-t)x + ty \mid 0 \leq t \leq 1\}.$$

We call x, y the end points of $\overline{xy}$. Given a point $x \in \mathbb{R}^n$, the **ray** (or half line) along x is the set

$$\mathbb{R}^+x := \{rx \mid r \in \mathbb{R}^+\}.$$

We can imagine that a ray is a kind of edge but with one of the end points at 'infinity', and so that '$\overline{x\infty}$' runs from x to ∞ like a light ray.

Definition 3.1. *An* **affine subspace** *of $\mathbb{R}^n$ is a subset A defined by a finite number of linear equations of the form*

$$a_1x_1 + \cdots + a_nx_n = b$$

where $a_1, .., a_n, b \in \mathbb{R}$ are given. Here $(x_1, .., x_n)$ denote the usual coordinates on $\mathbb{R}^n$. If all the defining equations of the set A are homogeneous, i.e. of the form $a_1x_1 + \cdots + a_nx_n = 0$, then we call A a **linear subspace** *of $\mathbb{R}^n$.*

Definition 3.2. *A nonempty subset Δ of $\mathbb{R}^n$ is called a* **hull** *if it contains the edge connecting any two points in Δ, i.e. if $\forall x, y \in \Delta$ then $\overline{xy} \subset \Delta$. In this case, its* **dimension** $\dim \Delta$ *is the dimension of the smallest* affine *subspace of $\mathbb{R}^n$ containing Δ. If this number is m then we call Δ an m-hull.*
A hull σ in $\mathbb{R}^n$ is called a **beam** *if it contains the ray along any vector in σ, i.e. if $\forall x \in \sigma$ then $\mathbb{R}^+x \subset \sigma$. We also require that the only linear subspace of $\mathbb{R}^n$ that σ contains is* (0).
Since a beam σ in $\mathbb{R}^n$ contains the zero vector 0, the smallest affine *subspace containing σ is also the smallest* linear *subspace containing σ. If the dimension of this linear subspace is m then we call σ an m-beam.*

Note that *every beam is a hull*, but a hull need not be a beam. It is clear that the intersection of hulls is a hull, and likewise for beams.

Example 3.3. *The hull of k points $v_1, ..., v_k \in \mathbb{R}^n$.*

Given such k points, what is the smallest hull containing those k points? First, this hull necessarily contains the set

$$\Delta \equiv \langle v_1, .., v_k\rangle := \{x_1v_1 + \cdots + x_kv_k \mid x_1, .., x_k \in \mathbb{R}^+, \ \sum_i x_i = 1\}.$$

(Why?) Let $x, y \in \Delta$, so we can write $x = \sum_i x_i v_i$ and $y = \sum_i y_i v_i$ with $x_i, y_i \in \mathbb{R}^+$ and $\sum_i x_i = \sum_i y_i = 1$. Then the points on $\overline{xy}$ are exactly of the form

$$(1-t)x + ty = \sum_i ((1-t)x_i + ty_i)\, v_i, \quad 0 \le t \le 1.$$

Since $(1-t)x_i + ty_i \in \mathbb{R}^+$ and $\sum_i ((1-t)x_i + ty_i) = (1-t) - t = 1$, this shows that $\overline{xy} \subset \Delta$. Hence the set Δ is a hull. This shows that Δ *is* the smallest hull containing the k points. Thus we call Δ the hull **generated** by the k points $v_1, .., v_k$.

This example also shows that if Δ is *any* hull and if $v_1, .., v_k \in \Delta$, then Δ must contain the hull $\langle v_1, .., v_k\rangle$. For since the latter is the *smallest* hull containing those k points.

Example 3.4. *The beam of k vectors $v_1, .., v_k \in \mathbb{R}^n$.*

Given such k vectors, what is the smallest beam containing those k vectors? Since this beam is itself a hull and must contain those k vectors, the beam necessarily contains the hull $\langle v_1, .., v_k\rangle$. But since it must also contain the ray along any point x in the hull, the beam must contain the set

$$\sigma \equiv \mathbb{R}^+\langle v_1, .., v_k\rangle \equiv \{x_1 v_1 + \cdots + x_k v_k \mid x_1, .., x_k \in \mathbb{R}^+\}.$$

We claim that σ is a beam. Let $x \in \sigma$, so we can write $x = \sum_i x_i v_i$ with $x_i \in \mathbb{R}^+$. Clearly if $r \in \mathbb{R}^+$ then $rx = \sum_i r x_i v_i \in \sigma$. This shows that $\mathbb{R}^+ x \subset \sigma$, hence the set σ is beam. This shows that σ *is* the smallest beam containing the k vectors. Thus we call σ beam **generated** by the k vectors $v_1, .., v_k$.

This example also shows that if σ is *any* beam and if $v_1, .., v_k \in \sigma$, then σ must contain the beam $\mathbb{R}^+\langle v_1, .., v_k\rangle$. For since the latter is the *smallest* beam containing those k points.

For example, if $v_1, v_2 \in \mathbb{R}^n$ are two distinct points, then their hull $\langle v_1, v_2\rangle$ is exactly the edge $\overline{v_1 v_2}$. If $v_1, v_2, v_3 \in \mathbb{R}^n$ are three pairwise distinct points, then their hull $\langle v_1, v_2, v_3\rangle$ is the triangle with those three points as vertices, and the three edges $\overline{v_1 v_2}$, $\overline{v_1 v_3}$ and $\overline{v_2 v_3}$. If v_1 is a nonzero vector, then the beam $\mathbb{R}^+\langle v_1\rangle$ is exactly the ray $\mathbb{R}^+ v_1$. If v_1, v_2 are two vectors which are not multiple of one another, then the beam $\mathbb{R}^+\langle v_1, v_2\rangle$ is the set 'wedged' in between the two rays $\mathbb{R}^+ v_1$ and $\mathbb{R}^+ v_2$. If $v_1, v_2, v_3 \in \mathbb{R}^n$ are three linearly independent vectors then their beam $\mathbb{R}^+\langle v_1, v_2, v_3\rangle$ is the set 'wedged' in between and the three beams $\mathbb{R}^+\langle v_1 v_2\rangle$, $\mathbb{R}^+\langle v_1 v_3\rangle$ and $\mathbb{R}^+\langle v_2 v_3\rangle$.

In this project, we will be studying certain collections of hulls and beams generated by **integral vectors**, i.e. vectors belonging in $\mathbb{Z}^n$. Thus from now on, *a hull or a beam is assumed to be generated by a finite number of integral vectors in $\mathbb{Z}^n$.* Observe that

if g is an automorphism of $\mathbb{Z}^n$, then g maps an integral hull to an integral hull, and an integral beam to an integral beam.

Definition 3.5. *Two hulls Δ, Δ' (resp. beams σ, σ') in $\mathbb{R}^n$ are* **equivalent***, if there is an automorphism $g \in \operatorname{Aut} \mathbb{Z}^n$ such that $g\Delta = \Delta'$ (resp. $g\sigma = \sigma'$).*

Definition 3.6. *If Δ is a hull, then Its* **dual hull** *$\Delta^\vee$ is the set*

$$\Delta^\vee := \{x \in \mathbb{R}^n \mid y \cdot x \geq -1, \quad \forall y \in \Delta\}.$$

If σ is a beam, then its **dual beam** *$\sigma^\vee$ is the set*

$$\sigma^\vee := \{x \in \mathbb{R}^n \mid y \cdot x \geq 0, \quad \forall y \in \sigma\}.$$

Exercise 3.7. *Verify that $\Delta^\vee$ and $\sigma^\vee$ above are respectively a hull and a beam in $\mathbb{R}^n$. Moreover, we have*

$$(\Delta^\vee)^\vee = \Delta, \quad (\sigma^\vee)^\vee = \sigma.$$

Exercise 3.8. *In $\mathbb{R}^2$ and $\mathbb{R}^3$, try to draw the pictures of $\Delta^\vee$ for various triangles Δ, and the pictures of $\sigma^\vee$ for various 'wedges' σ.*

Bounding hyperplanes. Observe that for a fixed vector $y = (y_1, .., y_n) \in \mathbb{R}^n$, the equation

$$y \cdot x = y_1 x_1 + \cdots + y_n x_n = -1$$

defines a **hyperplane**, an affine subspace of dimension $n-1$, which separates $\mathbb{R}^n$ into two half spaces. One of them being defined by the inequality

$$y \cdot x \geq -1$$

one for each $y \in \Delta$ which is 'normal' to the hyperplane. This shows that $\Delta^\vee$ is nothing but the intersection of a collection of half spaces. Hence $\Delta^\vee$ is a set bounded by a collection of bounding hyperplanes.

Exercise 3.9. *Show that if $\Delta = \langle v_1, .., v_k \rangle$ then*

$$\Delta^\vee = \cap_{i=1}^k \{x \in \mathbb{R}^n \mid v_i \cdot x \geq -1\}.$$

In other words, $\Delta^\vee$ is defined by k bounding hyperplanes specified by the k 'normal' vectors $v_1, .., v_k$.

It can be shown that if Δ is an n-hull in $\mathbb{R}^n$ generated by a finite number of points and if Δ contains 0 in its interior (i.e. the subset of points in Δ not on its bounding hyperplanes), then $\Delta^\vee$ is also an n-hull generated by a finite number of points and contains 0 in its interior. However, even if Δ integral, $\Delta^\vee$ need not be integral in general. (Example?) Likewise, if σ is an n-beam in $\mathbb{R}^n$ generated by a finite number of vectors, then $\sigma^\vee$ is also an n-beam generated by a finite number of vectors.

From now on, all hulls and beams in $\mathbb{R}^n$ are assumed to be generated by finitely many integral points or vectors in $\mathbb{R}^n$, and that a hull contains 0 in its interior.

Exercise 3.10. *Verify that if a beam in $\mathbb{R}^n$ is generated by a finite number of integral vectors, then we can choose generators of the dual beam $\sigma^\vee$ that are integral vectors.*

Definition 3.11. *An integral n-hull Δ in $\mathbb{R}^n$ is called* **perfect** *if its dual hull $\Delta^\vee$ is also integral.*

For example, it is easy to see that there is just one perfect 1-hull in $\mathbb{R}$, namely the interval $\Delta = [-1, 1]$ in $\mathbb{R}$. But it is much less obvious what a perfect 2-hull in $\mathbb{R}^2$ would be like, though there are some obvious examples.

Example 3.12. *Let Δ be the square with the vertices $(1, 1), (-1, 1), (-1, -1), (1, -1)$. You can check that $\Delta^\vee$ is also a square but with vertices $(1, 0), (0, 1), (-1, 0), (0, -1)$. Hence we found a pair of perfect 2-hulls.*

Exercise 3.13. *Verify that if Δ is a perfect hull, then $\Delta^\vee$ is also perfect. This shows that perfect hulls come in pairs!*

Exercise 3.14. *If Δ is a perfect hull, prove that the interior Δ° of Δ contains exactly one integral point, namely the origin 0.*

Here comes the problems in this project.

Problem 3.15. *Classify, up to equivalence, all perfect hulls in $\mathbb{R}^2$. That is, write down a complete list of such perfect hulls, no two of which are equivalent.*

Problem 3.16. *Can you do the same for perfect hulls in $\mathbb{R}^3$? Can you give an algorithm to generate a complete list?*

Problem 3.17. *For each $n = 2, 3, 4, ...,$ are there infinitely many or just finitely many inequivalent perfect hulls? Can you give an algorithm to generate a complete list for each n?*

Definition 3.18. *Let Δ be a hull. If H is a bounding hyperplane of Δ, we call $H \cap \Delta$ a* **wall** *of the hull Δ. Note that a wall of Δ is itself a hull (usually of lower dimension.) If a wall of Δ has dimension k, we call it a k-wall of Δ. By convention, we treat both the emptyset $\emptyset$ and Δ itself walls of Δ, even though neither one strictly satisfy the definition above.*
Let σ be a beam. If H is a bounding hyperplane of σ, we call $H \cap \sigma$ a **wall** *of the beam σ. Note that a wall of Δ is itself a beam (usually of lower dimension.) If a wall of σ has dimension k, we call it a k-wall of σ. By convention, we treat the set $\{0\}$ and σ itself both as walls of σ.*

Definition 3.19. *We call an n-beam σ in $\mathbb{R}^n$* **nice***, if*

$$\sigma = \mathbb{R}^+\langle v_1, .., v_n\rangle$$

for some $v_1, .., v_n \in \mathbb{Z}^n$ such that $\mathbb{Z}^n = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \equiv \mathbb{Z}(v)$. We call a k-beam τ in $\mathbb{R}^n$ **nice***, if it is a k-wall of a nice n-beam in $\mathbb{R}^n$.*

Exercise 3.20. *Suppose $(v_1, .., v_n)$ is an independent tuple of vectors in $\mathbb{Z}^n$. Show that $\sigma = \mathbb{R}^+\langle v_1, .., v_n\rangle$ is an n-beam whose k-walls are exactly those of the form $\mathbb{R}^+\langle v_{i_1}, .., v_{i_k}\rangle$, where $1 \le i_1 < \cdots < i_k \le n$. Therefore σ has exactly $\binom{n}{k}$ many k-walls.*

As we shall see in class, that an n-beam $\langle v_1, ..., v_n\rangle$ is nice iff the $n \times n$ matrix $[v_1, .., v_n]$ given by the n column integral vectors $v_1, .., v_n$ is invertible and whose inverse is also an integral matrix. This holds iff the determinant $\det[v_1, .., v_n]$ is ± 1. Similarly, a k-beam τ in $\mathbb{R}^n$ is a nice iff

$$\tau = \mathbb{R}^+\langle v_1, .., v_k\rangle$$

for some $v_1, ..., v_k \in \mathbb{Z}^n$ and there exists $v_{k+1}, .., v_n \in \mathbb{Z}^n$ such that $\langle v_1, .., v_n\rangle$ is a nice n-beam in $\mathbb{R}^n$.

Definition 3.21. *A* **star** *S is a finite nonempty collection of beams satisfying the following two properties:*
(S1) if $\sigma_1, \sigma_2 \in S$ then $\sigma_1 \cap \sigma_2$ is a wall of σ_1 and σ_2.
(S2) if $\sigma \in S$ then every wall of σ is also in S;
If S, S' are stars in $\mathbb{R}^n$, we say that they are equivalent if there exists $g \in \mathrm{Aut}\,\mathbb{Z}^n$ such that $gS = S'$. In other words, the map

$$S \to S', \quad \sigma \mapsto g\sigma$$

is bijective.
A star S is **nice** *if every beam in S is nice. The* **support** *of a star S is the set*

$$\mathrm{supp}\, S = \cup_{\sigma\in S}\sigma.$$

A star S in $\mathbb{R}^n$ is **complete** *if $\mathrm{supp}\, S = \mathbb{R}^n$.*
Let $\sigma_1, .., \sigma_k$ be beams in $\mathbb{R}^n$, and S be the set consisting those beams together with all their walls. If S is a star then we say that S is **generated** *by $\sigma_1, ..., \sigma_k$ and we write*

$$S = \langle \sigma_1, .., \sigma_k\rangle.$$

Problem 3.22. *Describe all stars in $\mathbb{R}^2$ generated by exactly two 2-beams. Which ones of them are inequivalent? Which ones of them are nice?*

Problem 3.23. *Classify, up to equivalence, all nice and complete stars in $\mathbb{R}^2$ having exactly three 2-beams.*

Problem 3.24. *Classify, up to equivalence, all nice and complete stars in $\mathbb{R}^2$ having exactly four 2-beams.*

Problem 3.25. *Classify, up to equivalence, all nice and complete stars in $\mathbb{R}^n$ having exactly three n-beams.*

4. More Projects?

In case you find the projects proposed here too easy for you, I want you to talk to me. I would be happy to help you find a project that meets your challenge!

HAPPY EXPLORING!