LINEAR ALGEBRA HOMEWORK 2 - DUE 7/25

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Assume U, V, W are F-vector spaces.

Exercise 0.1. *As in class, let* $\mathrm{Hom}(U, V) \equiv \mathrm{Hom}_F(U, V)$ *be the set of* F*-linear maps* $U \to V$*. Consider the MMC map*

$$\Phi : M_{k,l} \to \mathrm{Hom}(F^l, F^k), \quad A \mapsto L_A$$

Show that there is just one way to make $\mathrm{Hom}(F^l, F^k)$ *an* F*-space, such that* Φ *is linear. How would you make* $\mathrm{Hom}(U, V)$ *an* F*-vector space in general?*

Exercise 0.2. *If* $f : U \to V$ *and* $g : V \to W$ *are linear maps, verify that their composition* $gf \equiv g \circ f : U \to W$ *is also linear.*

Exercise 0.3. *Let* $\mathrm{Iso}(V, V)$ *be the set of isomorphisms (i.e. linear bijections)* $\phi : V \to V$*, and* $\mathrm{Iso}(V, W)$ *the set of isomorphisms* $f : V \to W$*. Suppose* $f_0 \in \mathrm{Iso}(V, W)$*. Show that the map*

$$T : \mathrm{Iso}(V, V) \to \mathrm{Iso}(V, W), \ \ \phi \mapsto f_0 \circ \phi$$

bijects.

Exercise 0.4. *Describe* $sol(E)$ *to the following system* E *in* $\mathbb{R}^4$*, by giving an isomorphism* $f : \mathbb{R}^k \to S_0$ *(including specifying the appropriate* k*):*

$$E : \quad \begin{aligned} x + y + z + t &= 0 \\ x + y + 2z + 2t &= 0 \\ x + y + 2z - t &= 0. \end{aligned}$$

Replace the 0 *on the right side of first equation by* 1*, and then describe the solution set* S_1 *of the new system. Find a 'nice' bijection (again as nice as you can think of)* $\mathbb{R}^k \to S_1$*. (Hint: Use a nice bijection* $S_0 \to S_1$*.)*

Exercise 0.5. WRITE UP *Let* E *be an* n*-variable* F*-linear system. Prove that*

$$\begin{aligned} & E \text{ is homogenous} \\ \Leftrightarrow \quad & 0 \in sol(E) \\ \Leftrightarrow \quad & sol(E) \text{ is } F\text{-subspace of } F^n. \end{aligned}$$

Try to make your proof as simple as possible, say less than half a page.

Exercise 0.6. *WRITE UP Let $F[x]_d$ be the F-subspace of $F[x]$ consisting of all polynomials $p(x)$ of degree at most d, i.e. the highest power x^n appearing in $p(x)$ is at most x^d. Consider the map*

$$L_n := (1-x^2)(\frac{d}{dx})^2 - 2x\frac{d}{dx} + n(n+1) : F[x]_d \to F[x]_d$$

for integer $n \geq 0$. Verify that L_n is F-linear. Describe $\ker L_n$ *by solving the linear equation*

$$L_n(f) = 0$$

for $n = 0, 1, 2$. Find $k \in \mathbb{Z}_{\geq 0}$ such that you can construct an F-isomorphism

$$f : \mathbb{R}^k \to \ker f.$$