LINEAR ALGEBRA HOMEWORK 3 – DUE 7/27

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F denotes a field. Assume U, V, W are F-vector spaces, and all dimensions are F-dimensions.

Exercise 0.1. *Let V be a F-subspace of F^n. Decide whether each of the following is TRUE of FALSE. Justify your answer. For (a)-(e), assume that* $\dim V = 3$.
(a) Any 4-tuple of V is linearly dependent.
(b) Any 2-tuple of V is linearly independent.
(c) Any 3-tuple of V is a basis.
(d) Some 3-tuple of V is a basis.
(e) V contains a linear subspace W with $\dim\ W = 2$.
(f) $(1, \pi)$, $(\pi, 1)$ *form a basis of* $\mathbb{R}^2$. *You can assume that* $|\pi - 3.14| < 0.01$.
(g) $(1, 0, 0)$, $(0, 1, 0)$ *do not form a basis of the plane* $x - y - z = 0$.
(h) $(1, 1, 0)$, $(1, 0, 1)$ *form a basis of the plane* $x - y - z = 0$.
(i) If A is a 3×4 matrix, then the subspace V of F^4 generated by the rows of A is at most 3 dimensional.
(j) If A is a 4×3 matrix, then the subspace V of F^3 generated by the rows of A is at most 3 dimensional.

Exercise 0.2. *WRITE UP Let*

$$V = \{(a + b, a, c, b + c) | a, b, c \in F\} \subset F^4.$$

Verify that V is an F-subspace of F^4. Find a basis of V.

Exercise 0.3. *WRITE UP Do this without using COD.*
(a) In 2 lines, show that any 3-tuple in $V = F^2$ is dependent. In 3 line, generalize this to any F-space V with $\dim V = n < +\infty$.
(b) Let $U \subset V$ be an F-subspace, and $\mathcal{S}$ the set of all independent tuples in U. In 5 lines, show that if $\dim V = n < +\infty$ then $\mathcal{S}$ contains a basis of U, and conclude that $\dim U \le \dim V$.
(Hint: Consider a tuple of **maximal** *length in $\mathcal{S}$; why does it exists?)*
(c) In 5 lines show that if furthermore $\dim U = \dim V$ then $U = V$.

Exercise 0.4. *Find a basis of sol(E) in F^4 for*

$$E: \quad x - y + 2z + t = 0.$$

Exercise 0.5. *Find a basis for each of the subspaces* $\ker L_A$ *and* $\operatorname{im} L_A$ *of* F^4, *where A is the matrix*

$$\begin{bmatrix} -2 & -3 & 4 & 1 \\ 0 & -2 & 4 & 2 \\ 1 & 0 & 1 & 1 \\ 3 & 4 & -5 & -1 \end{bmatrix}.$$

Exercise 0.6. *We know that $V^2 = V \times V$ form a vector space. Define an F-vector space structure on $U \times V$ a vector space. Let's call it the* **direct sum**

$U \oplus V$ of U, V. If $\dim U = k$ and $\dim V = n$, what is $\dim(U \oplus V)$? Prove your assertion in 5 lines.

Exercise 0.7. *WRITE UP We have the isomorphism $MTC : \mathrm{Hom}(F^2, V) \to V^2$. Now let's assume $\dim U = 2$. How would you write down an isomorphism*

$$\mathrm{Hom}(U, V) \to V^2 \quad ?$$

Can you describe the set of **all** *such isomorphisms?*
(Hint: Think about HW2 problem 3.)

Exercise 0.8. *(Revisit MMC) We specialize to the case $V = F^2$. Let $(v_1, v_2) \in V^2$, put $A = [v_1, v_2] \in M_{2,2}$, and write $v_1 = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix}$, $v_2 = \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix}$.*
(a) (A numerical test for isomorphism) Show that v_1, v_2 are 'parallel', i.e. one a scalar multiple of the other, iff they are dependent, iff

$$a_{11}a_{22} - a_{12}a_{21} = 0$$

iff L_A is not an isomorphism, iff (v_1, v_2) is not a basis of V.
(b) Now suppose L_A is an isomorphism. Can you find the matrix B corresponding to (under MMC) to the inverse isomorphism $L_A^{-1} : F^2 \to F^2$?