LINEAR ALGEBRA HOMEWORK 5 – FRIDAY 8/3

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Exercise 0.1. *Prove that for* $A \in M_{n,n}$
$$\det A^T = \det A.$$
Thus it doesn't matter whether we think of det *as a function of* n *row vectors or* n *column vectors.*
(Hint: Clearly understand the cases $n = 2, 3$ *first. Observe that* $Perm(3) \equiv S_3 \to S_3$, $\sigma \mapsto \sigma^{-1}$, *is a bijection.)*

Exercise 0.2. *Let* $x, y \in M_{n,n}$. *Recall that* x, y *are translates of each other iff there exists an invertible matrix* g *such that* $y = g^{-1}xg$. *Prove your assertions.*
(a) Suppose $\det x \neq \det y$. *Can* x, y *be translates of each other, i.e. can* $[x] = [y]$*?*
(b) Suppose $\det x = \det y$. *Does this imply that* $[x] = [y]$*?*

Exercise 0.3. *WRITE UP Let* $x_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, *and let* $[x_0]$ *be its translation class in* $M_{2,2}$. *Show that there is a surjective map*
$$\pi : [x_0] \to \mathbb{P}^1 := \textit{the set of all lines in } F^2$$
given by $x \mapsto \ker x$. *Here, a line in* F^2 *is a one dimensional subspace of* F^2. *Can you describe the subset*
$$\pi^{-1}(\ker x) = \{y \in [x_0] | \ker y = \ker x\}$$
for each x*? Prove your assertions.*

Exercise 0.4. *Let* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_{2,2}$, *and let* t *be a variable. Write down the polynomial function* $\det(A - tI) \in F[t]$. *What is its degree? What is the coefficient of the highest power of* t *and the lowest power of* t *for this polynomial? Generalize your answers to* $n \times n$ *matrices.*

Exercise 0.5. *WRITE UP Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in M_{2,2}(\mathbb{C})$. *Find an invertible matrix* B *such that* $B^{-1}AB$ *is upper triangular.*

Exercise 0.6. *WRITE UP Let* A *be an* F*-algebra and* V *be a finite dimensional* A*-space. Show that* V *is a quotient* A*-space of* V. *In other words, there exists a surjective* A*-space homomorphism*
$$A^{\oplus k} \twoheadrightarrow V$$
where $A^{\oplus k}$ *is the direct sum of* k *copies of* A, *regarded as an* A*-space. Show that if* A *is a semi-minimal as an* A*-space, then any* A*-space* V *is semi-minimal.*