2018 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

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0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). For simplicity, you can think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, \ldots$. Likewise a linear map means an F-linear, and a matrix means a matrix with entries in F. Put

$$
\begin{aligned}
\mathrm{Hom}(U,V) &:= \{\text{linear maps } U \to V\} \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \quad \text{the algebra of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V | f \text{ is bijective}\} \\
\mathrm{Aut}_n F &:= \mathrm{Aut}\, F^n \\
\mathrm{SAut}_n F &:= \{A \in \mathrm{Aut}_n F | \det A = 1\} \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n \text{ } F\text{-matrices} \\
& \text{with the usual matrix product} \\
I &\equiv I_n := [e_1, .., e_n], \quad \text{the identity matrix in } M_n \\
(M_n, *) &:= \text{the } * \text{ algebra of } n \times n \text{ matrices} \\
& \text{with } A * B := AB - BA \\
(SM_n, *) &:= \text{the } * \text{ algebra of } n \times n \text{ matrices } A \\
& \text{with } \mathrm{tr}\, A := a_{11} + \cdots + a_{nn} = 0
\end{aligned}
$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project 1

Consider the F-algebra M_n of $n \times n$ F-matrices, where the product is the usual matrix product. This project is about understanding this algebra and how it acts on F-vector spaces.

Let V be finite dimensional F-vector space. Just like M_n, the space of F-linear maps $\mathrm{End}\, V$ is also an F-algebra under composition.

Definition 1.1. *We call V an M_n-**space** if it is equipped with an F-algebra homomorphism $\rho : M_n \to \mathrm{End}\, V$, i.e. an F-linear map such that*

$$\rho(I_n) = 1_V, \quad \rho(xy) = \rho(x)\rho(y), \quad \forall x, y \in M_n.$$

*We sometimes write this M_n-space as a pair (V, ρ), and say that M_n **acts** on V by ρ. We shall also write*

$$xv \equiv \rho(x)v$$

if no confusion arises.
*We say two M_n-spaces $(V_1, \rho_1), (V_2, \rho_2)$ are **isomorphic** if there is an F-isomorphism $f : V_1 \to V_2$ that is **compatible** with the M_n-actions on the two vector spaces, i.e.*

$$f(\rho_1(x)v_1) = \rho_2(x)f(v_1), \quad x \in M_2, \quad v_1 \in V_1$$

or simply

$$f(xv) = xf(v).$$

Definition 1.2. *Let (V, ρ) be an M_n-space. An F-space $W \subset V$ is an M_n-**subspace** if it is **invariant** under the M_n-action, i.e.*

$$\rho(x)v \in W, \quad \forall x \in M_n, \quad v \in W.$$

*Note that this makes (W, ρ) itself an M_n-space, where we treat $\rho(x) \equiv \rho(x)|W \in \mathrm{End}\, W$. We say that a M_n-space $V \neq (0)$ is **minimal**, if it has exactly two M_n-subspaces W, namely V and (0). We say that an M_n-space is **semiminimal** if it is a direct sum of minimal M_n-subspaces.*

Exercise 1.3. *Show that if (V, ρ) is a minimal M_n-space, then it is generated by any nonzero vector in V. In other words, $W = M_n v$ for any nonzero $v \in W$.*

Problem 1.4.
(a) Describe all possible solutions to the matrix equation system

$$x_1^2 = x_1, \quad x_2^2 = x_2, \quad x_1x_2 = x_2x_1 = 0$$

in two variables in M_2, up to conjugation by Aut_2.
(b) Describe all those conjugation classes that satisfy the additional equation

$$x_1 + x_2 = I_2.$$

(c) Describe all possible two-sided ideals of M_2.
(d) Generalize (a)-(c) to M_n.

We saw in class that the algebra M_n itself is an M_n-space on which M_n acts by left multiplication. We also saw that an F-subspace $W \subset M_n$ is an M_n-subspace iff W is a left ideal of M_n.

Problem 1.5.
(a) Describe all possible left ideals I of M_2. Which ones of them are isomorphic to each other?
(b) Classify all minimal M_2-spaces V up to isomorphisms.
(c) Classify all M_2-spaces V up to isomorphisms.

Problem 1.6. *Generalize both Problems 1.4 and 1.5 to M_n-spaces for all n.*

2. Statements of Problems in Project 2

We start by making the identification under the MMC algebra isomorphism

$$M_n \equiv M_{n,n}(F) \equiv \mathrm{End}\, F^n := \mathrm{Hom}(F^n, F^n)$$

by treating $A \equiv L_A$. Using the matrix product in M_n, we can define a *new product*, which we call **the $*$-product** by

$$* : M_n \times M_n \to M_n, \quad (x, y) \mapsto x * y := xy - yx.$$

This project is to study this new algebra $(M_n, *)$, and how it operates on F-vector spaces in ways that are compatible with the $*$-product (much as in Project 1, we study the F-algebra $(M_n, \times)$ with the usual matrix product $\times$, and how it operators on vector spaces in ways that is compatible with this product, namely via ring homomorphisms $f : M_n \to \mathrm{End}\, V$.) *For simplicity, you may assume that* $F = \mathbb{C}$ *in this project.*

Note that $x*y$ 'measures' the failure of x, y to commute with each other. Unlike the matrix product, which is noncommutative, the star product is almost commutative: $x * y = -y * x$. Unlike the matrix product, which is associative: $(xy)z = x(yz)$, the new product is not. But the failure of associativity can be described easily as follows.

Proposition 2.1. *For $x, y, z \in M_n$, we have the* **$*$-identity**

$$\boxed{(x * y) * z + (y * z) * x + (z * x) * y = 0.} \tag{2.1}$$

By moving two terms to the other side and keeping track of sign changes, we can write this identity as

$$x * (y * z) - (x * y) * z = y * (x * z)$$

which describes how the $*$-product fails to be associative.

We can also rewrite the cyclic identity as

$$x * (y * z) - y * (x * z) = (x * y) * z.$$

We let M_n 'operate' on itself by defining the **operation map**

$$\rho : M_n \to \mathrm{End}\, M_n, \quad \rho(x)y := x * y \tag{2.2}$$

(which assigns to each $x \in M_n$ a linear operator on M_n), then the cyclic identity above becomes

$$\rho(x)\rho(y)z - \rho(y)\rho(x)z = \rho(x * y)z.$$

Since z is arbitrary, this become

$$\boxed{\rho(x) * \rho(y) = \rho(x * y).} \tag{2.3}$$

Note that the $*$-product on the left is defined on the space $\mathrm{End}\, M_n$, while the one on the right is on the space M_n. The identity connects the two $*$-products through the linear map $\rho : M_n \to \mathrm{End}\, M_n$ in such a way that is compatible with the two $*$-products. *In this project, we shall simply write*

$$\boxed{M_n \equiv (M_n, *)}$$

to mean the $*$-algebra M_n but equipped with the $*$-product. Now just like F^n, the set $M_n \equiv \mathrm{End}\, F^n$ is an F-vector space of dimension n^2 with the 'standard' basis $(e_{ij}) \equiv (e_{11}, e_{12}, ..., e_{nn})$. With this basis, we can also treat $\mathrm{End}\, M_n \equiv \mathrm{End}\, F^{n^2} \equiv M_{n^2}$. So, then $\rho(x) * \rho(y)$ above is nothing but the $*$-product on the matrix space M_{n^2}, and the cyclic identity relates the $*$-products on the two different matrix spaces M_n and M_{n^2}!

More generally, we can replace F^n by any any F-vector space V, and use the composition product on the algebra $E = \mathrm{End}\, V$ to define a $*$-product in exactly the same way:

$$* : E \times E \to E, \quad (x, y) \mapsto xy - yx.$$

This $*$-product clearly satisfies the $*$-identity as well. Again we write $\mathrm{End}\, V$ to denote the $*$-algebra $\mathrm{End}\, V$ equipped with this $*$-algebra structure. This motivates the following.

Definition 2.2. *A $F*$-algebra is an F-vector space E, equipped with an F-bilinear map*

$$* : E \times E \to E, \quad (x, y) \mapsto x * y$$

satisfying

$$x * y = -y * x$$

and the $*$-idenity (2.1).

We have seen that for any vector space V, $\mathrm{End}\, V$ is naturally a $*$-algebra.

Definition 2.3. *An M_n-space is an F-vector space V, equipped with a $*$-algebra homomorphism*

$$\rho : M_n \to \mathrm{End}\, V$$

That is to say, ρ is an F-linear map such that

$$\rho(x * y) = \rho(x) * \rho(y).$$

We often write such an M_n-space as a pair (V, ρ), and say that M_n acts on V through ρ.

Note that the $*$-identity (2.3) on M_n is simply the statement that the map

$$\rho : M_n \to \mathrm{End}\, M_n$$

given by (2.2) is an M_n-space homomorphism.

Next, introduce the **Kronecker's delta symbol** δ_{ab} to mean 0 if $a \neq b$ and 1 if $a = b$.

Proposition 2.4. *For $i, j, p, q = 1, .., n$,*

$$\boxed{(e_{ij})_{pq} = \delta_{pi}\delta_{qj}, \quad e_{ij}e_{kl} = e_{il}\delta_{jk}, \quad e_{ij} * e_{kl} = e_{il}\delta_{jk} - e_{kj}\delta_{li}.}$$

Exercise 2.5. *Let V be a vector space and $A, B, C \in \mathrm{End}\, V$. Show that the $*$-product satisfies the 'Leibniz rule'*

$$A * (BC) = (A * B)C + B(A * C).$$

Compute $A * B^n$ for $n \geq 0$.

Definition 2.6. *Let (V_1, ρ_1) and (V_2, ρ_2) be M_n-spaces. An M_n-homomorphism is a linear map $f : V_1 \to V_2$ that is compatible with the M_n action on V_1, V_2. Namely,*

$$f(\rho_1(x)v_1) = \rho_2(x)f(v_1), \quad x \in M_n, \ v_1 \in V_1.$$

An M_n-isomorphism is a bijective M_n-homomorphism.

Definition 2.7. *An F-subspace $L \subset M_n$ is a* $*$**-subalgebra** *of M_n if it satisfies*

$$x * y \in L, \quad \forall x, y \in L.$$

An F-subspace $J \subset M_n$ is a $*$**-ideal** *of M_n if it satisfies*

$$M_n * J \subset J.$$

Similarly, if E is a $*$-algebra, and V an E-space, the notions of invariant E-subspace of V, and of minimal E-space also be defined as in the case of F-algebras in Project 1 before.

Exercise 2.8. *Show that the subspace of zero trace matrices*

$$SM_n := \{A \in M_n | \operatorname{tr} A = a_{11} + \cdots + a_{nn} = 0\}$$

is a $*$-ideal of M_n, i.e.

$$M_n * SM_n \subset SM_n.$$

In particular, this also shows the space SM_n is itself a $*$-algebra, which we shall denote by SM_n. We can therefore define SM_n-spaces the same way we did for M_n-spaces.

Exercise 2.9. *Show that $SM_2 = F(h, x, y)$ where*

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Compute 'the table of $*$-products':

$$h * x, \quad h * y, \quad x * y.$$

Show that M_2 is the direct sum of two $*$-ideals:

$$M_2 = FI_2 \oplus SM_2.$$

Hence $I_2 * SM_2 = 0$, i.e. $I_2 x - x I_2 = 0$ for all $x \in SM_2$.

Exercise 2.10. *If $f : E \to E'$ is a $*$-algebra homomorphism, then $\ker f \subset E$ is a $*$-ideal, and $\operatorname{im} f \subset E'$ is a $*$-subalgebra.*

Problem 2.11. *Describe all $*$-ideals of the $*$-algebra M_n, for all n.*

Exercise 2.12. *Show that if $f : V_1 \to V_2$ is a nonzero homomorphism of two minimal SM_2-spaces, then is an isomorphism. Show that if $V_1 = V_2$, the automorphism f is a scalar multiple of the identity map. Is the statement still true if we replace SM_2 by the $*$-algebra M_2?*

Exercise 2.13. *Let (V, ρ) be an SM_2-space, and write $z \equiv \rho(z) \in \operatorname{End} V$ for $z \in SM_2 = F(h, x, y)$. Let $0 \neq v_0 \in V$ be an eigenvector of h, say*

$$hv_0 = \lambda v_0.$$

(a) Show that xv_0, yv_0 are also eigenvectors of h. Do the same for the vectors $y^p x^q v_0$, $p, q \geq 0$. Compute the corresponding eigenvalue in each case.
(b) Compute the operators (i.e. linear map on V) $y * x^2 := yx^2 - x^2 y$.

Exercise 2.14. *Let E be a $*$-algebra and (V, ρ) an E-space. Fix a basis $(v_1, ..., v_n)$ of V, and let $W = F[v_1, .., v_n]$ be the vector space of all polynomial expressions in the 'variables' $v_1, ..., v_n$. Show that there is a 'natural' action*

$$\rho' : E \to \operatorname{End} W$$

such that $\rho'(a)v_i = \rho(a)v_i$. Moreover, if $W_k \subset W$ is the (finite dimensional) subspace of homogeneous polynomial of degree k, then W_k is invariant under the E-action.

Problem 2.15. *Classify all 1-dimensional SM_2-spaces up to isomorphisms. Do the same for M_2-spaces.*

Problem 2.16. *Classify all minimal 2-dimensional SM_2-spaces up to isomorphisms. Do the same for M_2-spaces.*

Problem 2.17. *Classify all minimal k-dimensional SM_2-spaces up to isomorphisms, for all k.*

Problem 2.18. *Generalize Problems 2.15-2.16 to SM_3-spaces.*

Problem 2.19. *Generalize Problems 2.15-2.16 to SM_n-spaces, for all n.*