2019 MATHCAMP RESEARCH PROJECTS: LINEAR ALGEBRA

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This is a preliminary version of the research projects, subject to change later.

0. Basic Assumptions and Notations

Unless stated otherwise, we shall make the following assumptions and use the following notations. F will denote a field of characteristic zero (i.e. F contains $\mathbb{Z}$ as a subset). For simplicity, you can think about the case $F = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers. A vector space means a finite dimensional F-vector space, usually denoted by $U, V, W, ...$ Likewise a linear map means an F-linear, and an F-matrix means a matrix with entries in F. Put

$$\begin{aligned}
\mathrm{Hom}(U,V) &:= \{\text{linear maps } U \to V\} \\
\mathrm{End}\, V &:= \mathrm{Hom}(V,V), \text{ the algebra of linear maps } V \to V \\
\mathrm{Aut}\, V &:= \{f \in \mathrm{End}\, V | f \text{ is bijective}\} \\
\mathrm{Aut}\,_n F &:= \mathrm{Aut}\, F^n \\
(M_n, \times) \equiv M_n \equiv M_{n,n}(F) &:= \text{the associative algebra of } n \times n\ F\text{-matrices} \\
&\quad\ \text{with the usual matrix product} \\
I &\equiv I_n := [e_1, .., e_n], \text{ the identity matrix in } M_n
\end{aligned}$$

We usually denote composition of maps as $fg \equiv f \circ g$.

These objects will be quite thoroughly studied in class during the first two weeks.

1. Statements of Problems in Project 1

Consider the F-algebra M_n of $n \times n$ F-matrices, where the product is the usual matrix product. This project is about understanding this algebra and how it 'interacts' with abstract F-vector spaces.

Let V be finite dimensional F-vector space. Just like M_n, the space of F-linear maps $\mathrm{End}\, V$ is also an F-algebra under composition.

Definition 1.1. *We call V an* M_n**-space** *if it is equipped with an F-algebra homomorphism $\rho : M_n \to \mathrm{End}\, V$, i.e. an F-linear map such that*

$$\rho(I_n) = 1_V, \quad \rho(xy) = \rho(x)\rho(y), \quad \forall x, y \in M_n.$$

We sometimes write this M_n-space as a pair (V, ρ), and say that M_n **acts** *on V by ρ. We shall also write*

$$xv \equiv \rho(x)v$$

if no confusion arises.
We say two M_n-spaces $(V_1, \rho_1), (V_2, \rho_2)$ are **isomorphic** *if there is an F-isomorphism $f : V_1 \to V_2$ that is* **compatible** *with the M_n-actions on the two vector spaces, i.e.*

$$f(\rho_1(x)v_1) = \rho_2(x)f(v_1), \quad x \in M_2, \quad v_1 \in V_1$$

or simply

$$f(xv) = xf(v).$$

Definition 1.2. *Let (V, ρ) be an M_n-space. An F-space $W \subset V$ is an* M_n**-subspace** *if it is* **invariant** *under the M_n-action, i.e.*

$$\rho(x)v \in W, \quad \forall x \in M_n, \quad v \in W.$$

Note that this makes (W, ρ) itself an M_n-space, where we treat $\rho(x) \equiv \rho(x)|W \in \mathrm{End}\, W$. We say that a M_n-space $V \neq (0)$ is **minimal***, if it has exactly two M_n-subspaces W, namely V and (0). We say that an M_n-space is* **semiminimal** *if it is a direct sum of minimal M_n-subspaces.*

Exercise 1.3. *Show that if (V, ρ) is a minimal M_n-space, then it is generated by any nonzero vector in V. In other words, $W = M_n v$ for any nonzero $v \in W$.*

Problem 1.4.
(a) Describe all possible solutions to the matrix equation system

$$x_1^2 = x_1, \quad x_2^2 = x_2, \quad x_1x_2 = x_2x_1 = 0$$

in two variables in M_2, up to conjugation by Aut_2.
(b) Describe all those conjugation classes that satisfy the additional equation

$$x_1 + x_2 = I_2.$$

(c) Describe all possible two-sided ideals of M_2.
(d) Generalize (a)-(c) to M_n.

We saw in class that the algebra M_n itself is an M_n-space on which M_n acts by left multiplication. We also saw that an F-subspace $W \subset M_n$ is an M_n-subspace iff W is a left ideal of M_n.

Problem 1.5.
(a) Describe all possible left ideals I of M_2. Which ones of them are isomorphic to each other?
(b) Classify all minimal M_2-spaces V up to isomorphisms.
(c) Classify all M_2-spaces V up to isomorphisms.

Problem 1.6. *Generalize both Problems 1.4 and 2.2 to M_n-spaces for all n.*

2. Statements of Problems in Project 2

All graphs are assumed finite (i.e. have finite number of vertices), planar (i.e. you can draw on the plane) and oriented (i.e. every edge has a direction). All F vector spaces considered here are finite dimensional. A face of a graph L is a free region (i.e. no edges crosses it) bounded by edges of L. Let L be a graph, and $\mathcal{V}_L$, $\mathcal{E}_L$, $\mathcal{F}_L$ be the sets of vertices, edges, and faces of L respectively. We drop L if there is no confusion.

Every face $\phi \in \mathcal{F}$ is given the counter-clockwise orientation. Note that we can label ϕ by giving the list of edges in counterclockwise order. This induces an orientation on each edge e of ϕ which may or may not agree with the *given* orientation of e. We assign $+1$ to e and write $(\pm)_{\phi,e} = +1$ if the orientation induced by ϕ on e is the same as the given orientation of e. Otherwise we assign $(\pm)_{\phi,e} = -1$. We can label each $e \in \mathcal{E}$ by its vertices, say (a, b), where a is the initial and b is the end vertices of e. For convenience, we treat $(b, a) \equiv -(a, b)$ for each oriented edge $(a, b) \in \mathcal{E}$.

This project is about studying certain connections between graphs and F-vector spaces. Given such a graph L, we shall form a collection of vector spaces as follows:

- $C^0(L)$ is the F vector space given by taking the set of vertices of L as a basis of $C^0(L)$, i.e. $C^0(L) = F\mathcal{V}$.
- $C^1(L)$ is the F vector space given by taking the set of edges of L as a basis of $C^1(L)$, i.e. $C^1(L) = F\mathcal{E}$.
- $C^2(L)$ is the F vector space given by taking the set of faces of L as a basis of $C^2(L)$, i.e. $C^2(L) = F\mathcal{F}$.

This collection $C^\bullet(L)$ of vector spaces is called the **complex** of L.

We define a collection of linear maps, we call the boundary maps of L, between these vector spaces as follows. Write $C^i = C^i(L)$. Define

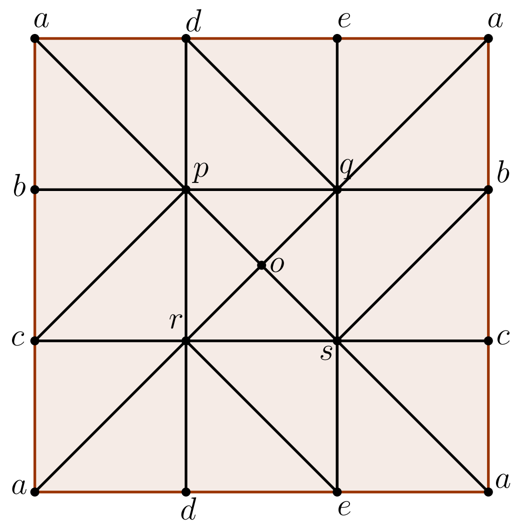
- $\partial_3 : (0) \to C^2$ is the zero map.
- $\partial_2 : C^2 \to C^1$, $\partial\phi = \sum(\pm)_{\phi,e} e$, where the sum is over all edges of ϕ.
- $\partial_1 : C^1 \to C^0$, $\partial(a, b) = a - b$.
- $\partial_0 : C^0 \to (0)$ is the zero map.

If there is no confusion, we shall drop all subscript from ∂.

Exercise 2.1. *Convince yourself that $\partial^2 = 0$, i.e. $\partial_i\partial_{i+1} = 0$ for all i. Therefore, we can define the* **homology** *spaces of L as*

$$H^i(L) := \ker \partial_i / \operatorname{im} \partial_{i+1}, \quad i = 0.1.2.$$

In this project, you are asked to determine the dimensions of the homology spaces of certain graphs. Consider the following graph, denoted by L.

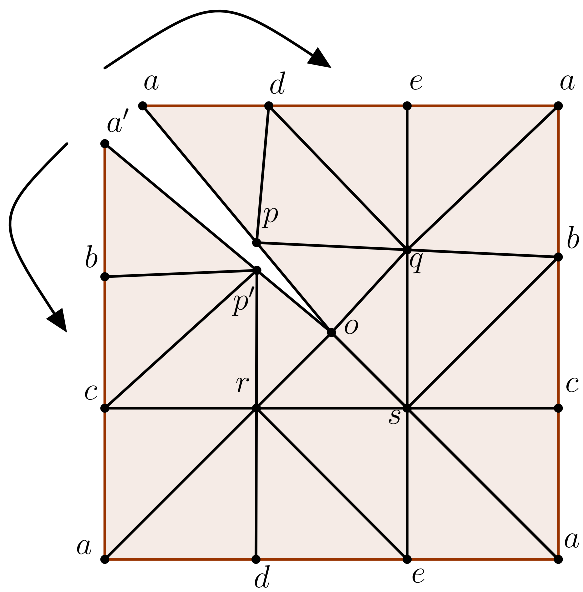


The boundaries of L (the brown edges) are identified via the labeling. We pick an orientation for every edge. For example, we can use the lexicographic ordering the vertex labels to give a edge an orientation: the edge with label (a, d) will have direction a to d, because a comes before d in the lexicographic ordering.

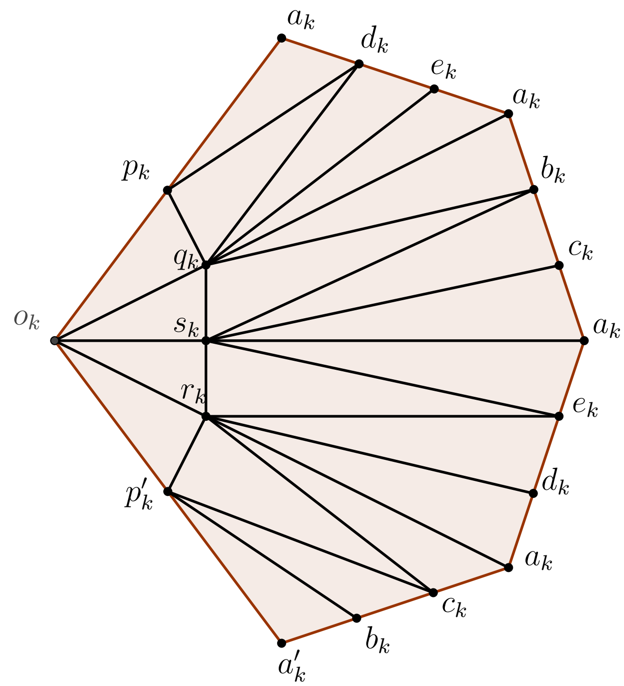
Problem 2.2. *Verify that*

- $\mathrm{H}_0(L) \simeq F$,
- $\mathrm{H}_1(L) \simeq F^2$,
- $\mathrm{H}_2(L) \simeq F$.

Next, we shall construct more complicated graphs by 'gluing' together copies of the graph L above. First, we 'cut open' L along the line segment $\overline{opa}$ in L, and then relabel the vertices as follows to get a new graph A_k.



Let A_k be the graph obtained by 'opening up' the shape along the arrows, and then assign the following new labels on the vertices.



For $n \in \mathbb{N}$, consider the set of graphs $\{A_1, \ldots, A_n\}$. Let S_n be the graph obtained by identifying $\overline{o_k p_k a_k}$ with $\overline{o_{k-1} p'_{k-1} a'_{k-1}}$ for $1 \leq k \leq n$. Here $\overline{o_0 p'_0 a'_0}$ is understood as $\overline{o_n p'_n a'_n}$. We orient each face in S_n counterclockwise and each edge in S_n, by lexicographic ordering as before. Again we get $C^{\bullet}(S_n)$ of the graph S_n. Note that $S_1 = L$.

Problem 2.3. *Find the homology spaces* $\mathrm{H}_q(S_n)$, *for all* n.

You should prove your answer.

3. Statements of Problems in Project 3

All vector spaces in this project are assumed finite dimensional.

A **complex** is a sequence of F-linear maps

$$\partial_{i+1} : C^{i+1} \to C^i, \quad i = 0, ..., d-1$$

such that the successive compositions are all zero: $\partial_i \partial_{i+1} = 0$. For convenience, we shall always assume that $C^i := (0)$, and that $\partial_{i+1} : C^{i+1} \to C^i$ are zero for all $i < 0$ or $i > d-1$. We denote the complex by $(C^\bullet, \partial)$ or simply $C^\bullet$, if ∂ is clear. The ith **homology** space of $C^\bullet$ is defined as the quotient spaces

$$H_i(C^\bullet) := \ker \partial_i / \operatorname{im} \partial_{i+1}, \quad i \in \mathbb{Z}.$$

A chain map between two complexes $(C^\bullet, \partial)$ and $(D^\bullet, \delta)$ is a collection of linear maps

$$f_i : C^i \to D^i$$

such that $\delta_i f_i = f_{i-1} d_i$ for all i. We shall denote the chain map by $f_\bullet : C^\bullet \to D^\bullet$.

We say that the two complexes are **equivalent** if there is a chain map $f_\bullet$ as above, such that each f_i is linear and bijective. We say that the two complexes are **quasi-equivalent** if there are two chain maps $f_\bullet : C^\bullet \to D^\bullet$, $g_\bullet : D^\bullet \to C^\bullet$, not necessarily bijective, such that their induced maps $\bar{f}_\bullet$, $\bar{g}_\bullet$ (to be defined in class) on homology are both linear and bijective, and are inverses of one another.

Problem 3.1. *Classify all equivalence classes of complexes $C^\bullet$ with at most two terms (i.e. all but two C^i are zero spaces). Do the same for 3-term complexes.*

Problem 3.2. *Classify all equivalence classes of complexes $C^\bullet$.*

Problem 3.3. *Classify all quasi-equivalence classes of complexes $C^\bullet$.*