ZETA FUNCTIONS, RATIONAL POINTS, AND THE WEIL CONJECTURE

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In his works on the quadratic reciprocity law, Gauss introduced the so-called Gauss sums. In order to compute these sums, Gauss was lead to evaluate the number of solutions of congruences of the following types:

$$ax^3 - by^3 \equiv 1 \mod p, \quad ax^4 - by^4 \equiv 1 \mod p, \quad y^2 \equiv ax^4 - b \mod p.$$

Weil studied this problem for the more general equation

$$a_0 x_0^{k_0} + \cdots + a_r x_r^{k_r} \equiv b \mod p. \tag{1}$$

The number of solutions is encoded in the ζ-function for the algebraic variety defined by this equation. Weil conjectured that the ζ-function of a smooth projective variety could be expressed in terms of the actions of the Frobenius correspondence on the cohomology groups of the algebraic variety, and the ζ-function should satisfy the Riemann hypothesis. These properties of the ζ-function give rise to the optimal estimate for number of rational points of the algebraic variety. The Weil conjecture was later proved by Grothendieck and Deligne using the ℓ-adic cohomology theory. In this course, we will express the ζ-function of (1) in terms of Gauss sums, and verify the Weil conjecture directly.

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