Probability and Statistics
Tsinghua Math Camp Prof. Paul Horn
Homework 6: Due Friday 8/4/2017

Note: You should get to know your coaches, Gavin St. John (`gavin.stjohn@du.edu`) and Zhe Liu (刘喆) (`z-liu16@mails.tsinghua.edu.cn`). If you need to contact me, my email address is `paul.horn@du.edu` or you may ask via WeChat.

I want very much to learn all of your names, but please be patient with me as I am not so good at pronouncing Chinese and I am very bad at remembering names (even non-Chinese names). So please help me by telling me your name very slowly and being patient with me as I try to learn and pronounce it.

Homework problems: To be turned in:

1. $n+1$ players play a game. Each person is independently a winner with probability p. The winners share a total prize of one unit – for instance if three people win, each wins $\frac{1}{3}$, and if no one wins, no one wins anything. Let A denote one of the specified players, and let X denote the amount that is received by A.

   - Compute $\mathbb{E}[X]$. (Hint: Let W be the number of winners. Compute $\mathbb{E}[X|W]$ as a function of W and then $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|W]]$)

2. The conditional variance of a random variable X given a random variable Y is

$$\mathrm{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y].$$

   Show the conditional variance formula:

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(X|Y).$$

   (This isn't too bad, write down the definition – but be careful: $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$, but $\mathbb{E}[\mathbb{E}[X|Y]^2] \neq \mathbb{E}[X]^2$!.

3. Give an example of a distribution so that $\mathbb{E}[X^n]$ exists and is finite for all $n \geq 0$, but so that $\mathbb{E}[e^{tX}]$ is infinite for any $t \geq 0$.
   Note: For a harder version of this (optional!) give an example of two different distributions which have the same expectations (that is $\mathbb{E}[X^n] = \mathbb{E}[Y^n] < \infty$ for all $n \geq 0$.) Because of results discussed in class, their expectations must grow quickly enough for $\mathbb{E}[e^{tX}]$ to be infinite for any $t > 0$.

4. Suppose X_i is Bernoulli with expectation $\frac{1}{i}$, for $1 \leq i \leq n$ and the X_i are independent. Then let $S_n = \sum_{i=1}^{n} X_i$.

   (a) Determine the expectation container $\phi_{S_n}(t)$ as a product of expectation containers.

(b) **Harder:** Note that $\mathbb{E}[S_n] = \sum_{i=1}^n \frac{1}{i} = H_n$. (We know $H_n \approx \ln n$, but is not exactly $\ln n$. The variance of S_n is $\sigma^2 = \sum \frac{1}{i}(1 - \frac{1}{i}) = H_n - \sum_{i=1}^n \frac{1}{i^2}$. Show that

$$\frac{S_n - H_n}{\sigma}$$

d-converge to a standard normal (that is $N(0,1)$) random variable?

(**Note:** I correctly normalized it to have mean zero and variance one, but it is *not* the sum of independent *identically distributed* random variables as the different X_i have different distributions. To do this, check: does $\phi_{\frac{S_n - H_n}{\sigma}}(t) \to e^{t^2/2}$?)
Hint: This is a bit hard. Here's one way to do it: Write out each of the expectation containers of the $\frac{X_i - \frac{1}{i}}{\sigma^2}$ by expanding the e^t terms as a Taylor series. Now try to multiply them all out, but remember that you are taking $n \to \infty$ so try and figure out what terms will remain, and which will dissapear when you take $n \to \infty$.