Probability and Statistics
Tsinghua Math Camp Prof. Paul Horn
Homework 7: Due Wednesday 8/9/2017

Note: You should get to know your coaches, Gavin St. John (gavin.stjohn@du.edu) and Zhe Liu (刘喆) (z-liu16@mails.tsinghua.edu.cn). If you need to contact me, my email address is paul.horn@du.edu or you may ask via WeChat.

I want very much to learn all of your names, but please be patient with me as I am not so good at pronouncing Chinese and I am very bad at remembering names (even non-Chinese names). So please help me by telling me your name very slowly and being patient with me as I try to learn and pronounce it.

Homework problems: To be turned in:

1. Suppose $X_1, \ldots, X_n$ are from a Uniform distribution on $(0, \theta)$. Find the MLE for θ, and an estimator for theta based on $\bar{X}$. Are your estimators unbiased? If not, can they be made unbiased by multiplying by a constant based on n? What constant?

2. Suppose X is a random variable (continuous or discrete) so that $\mathbb{E}[X^2]$ exists. Find the value of c that minimizes $\mathbb{E}[(X-c)^2]$.

3. Suppose $X_1, \ldots, X_n$ are an independent sample from a distribution $f(x;\theta) = \frac{\theta+1}{x}^{-\theta}$ for $1 \le x \le \infty$ where θ is unknown. Find an MLE for θ.

4. Suppose $X_1, X_2, \ldots$ are independent $N(\mu, \sigma^2)$ random variables, with unknown μ, and σ^2. Let

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X - \bar{X}_n)^2$$

denote the sample variance. Here $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.) Show that

   (a) $\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i - \bar{X}}{\sigma}\right)^2$

   (b) $\bar{X}_n$ and S_n^2 are independent.

Note: Prove this by induction on n. For normal random variables, $Cov(X,Y) = 0$ is enough for X and Y to be independent (you may assume this). For $n = 2$, the first fact follows as

$$\frac{S_2^2}{\sigma^2} = \frac{1}{2}\frac{(X_1 - X_2)^2}{\sigma^2},$$

and as X_1 and X_2 are independent $N(\mu, \sigma^2)$, $X_1 - X_2$ is $N(0, 2\sigma^2)$ so that $\frac{X_1 - X_2}{\sqrt{2}\sigma}$ is $N(0,1)$. Thus S_2^2/σ^2 is the square of a standard normal random variable and is $\chi^2(1)$. For the independence, note that $(X_1 + X_2)$ and $(X_1 - X_2)$ are independent because

$$\mathbb{E}[(X_1 + X_2)(X_1 - X_2)] = \mathbb{E}[X_1^2] - \mathbb{E}[X_2^2] = (\mathbb{E}[X_1] + \mathbb{E}[X_2])(\mathbb{E}[X_1] - \mathbb{E}[X_2]),$$

so that the covariance is zero and they are actually independent. Now try to write S_{n+1}^2 in terms of S_n^2 and an (independent) normal random variable (perhaps in terms of $\bar{X}_n$ and X_{n+1}) and do a similar trick of computing the covariance to show continued independence.